Local equations defining stable map moduli, arbitrary singularities, and resolution

Yi Hu

University of Arizona, USA

I will explain the matrix local equations defining the moduli spaces of stable maps of arbitrary genus, found jointly by Jun Li and the speaker. These equations already guided us to find explicit global resolutions for these moduli spaces in the cases when the genera are one (this follows Vakil and Zinger), and two. By Murphy's law, stable map moduli possess arbitrary singularities. Turning to this, I will explain Lafforgue's version of Mnev's universality, how it leads to standard local equations for arbitrary singularity types, and how it should guide to resolve arbitrary singularities.

Local equations for $\overline{M}_g(d, \mathbb{P}^n)$

Since a stable map $[u, C] \in \overline{M}_g(d, \mathbb{P}^n)$ is given by

$$u = [u_0, \ldots, u_n] : C \longrightarrow \mathbb{P}_n, \quad u_i \in H^0(u^* \mathscr{O}_{\mathbb{P}^n}(1))),$$

its deformation is determined by

the combined deformation of the curve C and the sections $\{u_i\}$.

Since moduli spaces of curves are smooth,

the singularity of $\overline{M}_g(d, \mathbb{P}^n)$ is caused by

the non-locally freeness of the direct image sheaf $\pi_* f^* \mathscr{O}_{\mathbb{P}^n}(1))$

of the universal family

$$\begin{array}{ccc} \overline{\mathcal{X}}_g(d, \mathbb{P}^n) & \xrightarrow{\quad f \quad} & \mathbb{P}^n \\ \downarrow \pi & & \\ [u, C] \in \overline{M}_g(d, \mathbb{P}^n) & & \end{array}$$

Focusing on the object $\mathbf{R}\pi_* f^* \mathscr{O}_{\mathbb{P}^n}(1))$ is a correct way to approach. It is two-term perfect.

To study the non-locally freeness of the direct image sheaf, by assigning to each stable map $[u, C]$ the divisor $D = u^{-1}(0) \subset C$, locally we can view $\overline{M}_g(d, \mathbb{P}^n)$ as a stack over the Artin stack $\mathcal{D}_g$ of pairs

$$\mathcal{D}_g = \{(C, D) \mid \text{ genus } g \text{ nodal curves } C \text{ and effective divisors } D \subset C\}$$

Over each chart $\mathcal{V} \subset \mathcal{D}_g$, by picking an auxiliary section of the universal curve $\rho : \mathcal{C} \longrightarrow \mathcal{V}$, we construct explicitly a two-term complex

$$\mathscr{R}^\bullet = [\mathscr{O}_{\mathcal{V}}^{\oplus(d+1)} \xrightarrow{\varphi} \mathscr{O}_{\mathcal{V}}]$$

whose sheaf cohomology gives the cohomology $\mathbf{R}\pi_* f^* \mathscr{O}_{\mathbb{P}^n}(1))$.

We then apply the deformation theory of nodal curves to derive a simple explicit form of the homomorphism φ in $\mathscr{R}^\bullet$. Under a suitable trivialization, we obtain

$$\varphi = (0, \xi_1, ..., \xi_d),$$

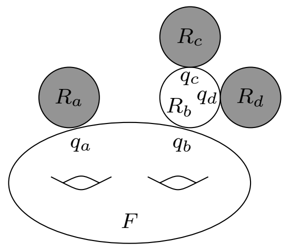
where each ξ_i is a suitable product $\prod_j \zeta_j$ of the pull back of regular functions whose vanishing loci are irreducible components of the set of nodal curves in $\mathcal{D}_g$.

Theorem. (Hu-Li) $\overline{M}_g(d, \mathbb{P}^n)$ is locally defined by:

$$[\varphi] \cdot \mathbf{w}^j = 0, \quad 1 \leq j \leq n$$

where $\mathbf{w} = (0, w_1^j, \cdots, w_d^j)$ with $w_i^j \in \mathbb{A}^1$.

Two examples



If $g(F) = 1$, then

$$\varphi = [0, \zeta_a, \zeta_b\zeta_c, \zeta_b\zeta_d, \cdots].$$

The local equation $[u, C]$ is

$$\zeta_a w_a^i + \zeta_b\zeta_c w_c^i + \zeta_b\zeta_d w_d^i = 0,\ w_a^i, w_c^i, w_d^i \in \mathbb{A}^1,\ 1 \le i \le n,$$

in $\mathcal{V} \times \mathbb{A}^{3n}$. (Other free variables are discared.)

If $g(F) = 2$, then

$$\varphi = \begin{bmatrix} 0 & a_{11}\zeta_a & c_{11}\zeta_b\zeta_c & d_{11}\zeta_b\zeta_d & \cdots \\ 0 & a_{21}\zeta_a & c_{21}\zeta_b\zeta_c & d_{21}\zeta_b\zeta_d & \cdots \end{bmatrix}$$

The local equation $[u, C]$ is

$$\begin{bmatrix} a_{11}\zeta_a & c_{11}\zeta_b\zeta_c & d_{11}\zeta_b\zeta_d \\ a_{21}\zeta_a & c_{21}\zeta_b\zeta_c & d_{21}\zeta_b\zeta_d \end{bmatrix} \begin{bmatrix} w_a^i \\ w_c^i \\ w_d^i \end{bmatrix},\ 1 \le i \le n$$

in $\mathcal{V} \times \mathbb{A}^{3n}$. (Other free variables are discared.)

Resolution: $g = 1$

Blowing up

$$\zeta_a = \zeta_b = 0,$$

this is the locus of curves with 2 rational tails, followed by blowing up

$$\zeta_a = \zeta_c = \zeta_d = 0,$$

this is the locus of curves with 23 rational tails. One calculates and finds that the singularity is resolved.

So, from local equations to global blowups, we obtain algebro-geometric version of Vakil-Zinger's analytic blowups

Theorem. (Vakil-Zinger, Hu-Li) **Blowing up $\overline{M}_1(d, \mathbb{P}^n)$ along the loci of curves with $2, 3, \cdots d$ rational tails, successively, we obtain a resolution of $\overline{M}_1(d, \mathbb{P}^n)$, in the sense that all irreducible components become smooth and meeting transversally.**

Resolution: $g = 2$

The sitation is more complex.

If a and b are not conjugate and there are no Weierstrass points, then, blowing up

$$\zeta_a = \zeta_b = 0,$$

followed by blowing up

$$\zeta_a = \zeta_c = \zeta_d = 0,$$

One calculates and finds that the singularity is resolved.

If, for example, a and b are also conjugate, we then further blow up

the locus where a and b are also conjugate.

One calculates and finds that the singularity is resolved.

Again, from local equations to global blowups, we obtain

Theorem. (Hu-Li-Niu) **Blowing up $\overline{M}_2(d, \mathbb{P}^n)$ along the loci of curves with $2, 3, \cdots d$ rational tails, successively, followed by additional blowups invloving the exceptional divisors from the previous round, Weierstrass and conjugate loci, we obtain a resolution of $\overline{M}_2(d, \mathbb{P}^n)$, in the sense that all irreducible components become smooth and meeting transversally.**

$\overline{M}_g(d, \mathbb{P}^n)$ possessess arbitrary singularities when g, d, n vary.

But, there are more classic models for singularity types.

Mnëv's universality

Just consider n points on $\mathbb{P}^2$:

$$p_1,\ p_2,\ p_3,\ \cdots,\ p_n \ \text{ in } \ \mathbb{P}^2,$$

and we want them to be in some fixed relative linear positions: that is, some points are required to be co-line. Three points in general linear position, they span $\mathbb{P}^2$; co-line, they span a line.

It is convenient to work with linear algebra, so we lift

$$v_1,\ v_2,\ v_3,\ \cdots,\ v_n \ \text{ in } \ \mathbf{k}^3,$$

and for any $I \subset [n]$, we let

$$d_I = \dim_{\mathbf{k}} \ \text{span}\{v_i \mid i \in I\}.$$

The matroid is a way to record the above.

A family $\underline{\mathrm{d}} = (d_I)_{I\subset[n]}$ of nonnegative integers $d_I \in \mathbb{N}$ verifying

- $d_\emptyset = 0, d_{[n]} = 3,$
- $d_I + d_J \le d_{I\cup J} + d_{I\cap J}$, for all $I, J \subset [n]$.

is called a matroid of rank 3 on the set $[n]$, where $[n] = \{1, \cdots, n\}$.

$$C_{\underline{\mathrm{d}}} = \{(p_1, p_2, \cdots, p_n) \in (\mathbb{P}^2)^n \mid \text{ their matroids are equal to the fixed } \underline{\mathrm{d}}\}.$$

Mnëv observes that $C_{\underline{\mathrm{d}}}$ contains arbitrary singularities over $\mathbb{Z}$.

Lafforgue's schematic proofs

An affine scheme X is defined by a finite number of polynomial equations with coefficients in $\mathbb{Z}$. Write these equations in the form of

$$P = Q$$

where P and Q are polynomials in $X_0, ..., X_k$ with positive integer coefficients. Let us then represent the expression of the polynomials P, Q as a function of $X_0, ..., X_k$ by introducing a number of additional variables $X_{k+1}, ..., X_m$ and and by imposing a number of equations of the form

$$X_\gamma = X_\alpha X_\beta$$

or

$$X_\gamma = X_\alpha + X_\beta,$$

or

$$X_\gamma = X_\alpha + 1.$$

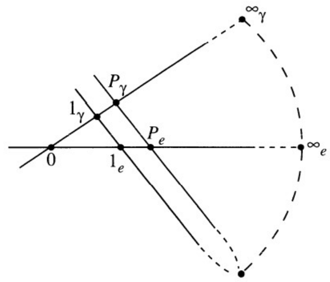


FIGURE 1.4.

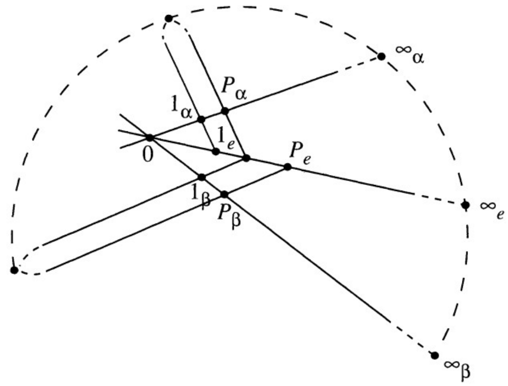


FIGURE 1.5.

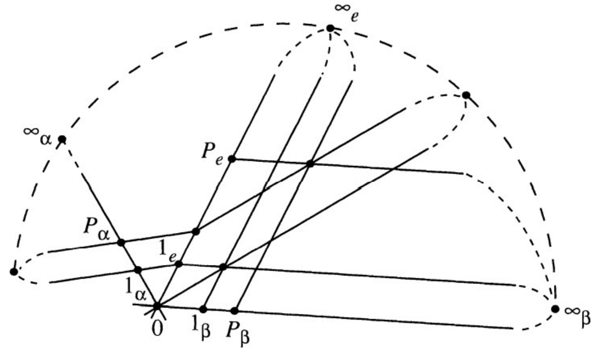


FIGURE 1.6.

after the other in a generic way, there are no other alignment relations besides the ones we have specified and therefore no other relations besides the equations (e).

We have defined a certain configuration space $C_S^{3,n}$. The transition to its quotient $\overline{C}_S^{3,n}$ by the free action of PGL_3 is to forget the choice of the origin 0, the first two

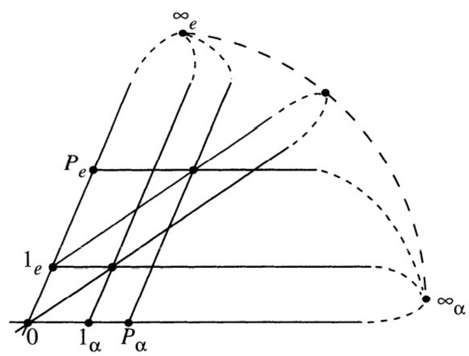


FIGURE 1.7.

points at infinity ∞_0 and ∞_1 defining the line ∞ and the two base points 1_0 and 1_1 on the lines $(0\infty_0)$ and $(1\infty_1)$. On the other hand, the choice of all the other points ∞_α and ∞_e on the line ∞ and 1_α, 1_e on the lines $(0\infty_\alpha)$ and $(0\infty_e)$ is equivalent to the introduction of as many additional affine variables in $\mathbb{A}^1$.

Thus the configuration space $\overline{C}_S^{3,n}$ is naturally isomorphic to an open subset of a product $X \times \mathbb{A}^N$.

The projection $U \to X$ is surjective because for any point of X with coordinates $Y_1, \ldots, Y_k$ and for generic T, all the $X_0, \ldots, X_m$ (related to each other by the equations (e) and to the $Y_1, \ldots, Y_k$ by $X_0 = T$, $X_1 = Y_1 + T$, ..., $X_k = Y_k + T$) satisfying

$$X_\alpha \neq 0,\ X_\alpha \neq 1,\ \ 0 \leq \alpha \leq m.$$

This results from the fact that the polynomial expressions of X_α, $k < \alpha \leq m$, as function of $X_0, \ldots, X_k$ each contain a unique monomial maximum total degree ≥ 1 and that this one is assigned the coefficient 1. □

It follows from Mnëv's theorem that configuration spaces $\overline{C}_S^{3,n}$ and thus also thin Schubert cells $\mathrm{Gr}_S^{3,E}$ classifying subspaces of dimension 3 of graded spaces $E = E_0 \oplus \cdots \oplus E_n$ have arbitrary singularities when n is allowed to be arbitrarily large. It is *a fortiori* the same for $\overline{C}_S^{r,n}$ and $\mathrm{Gr}_S^{r,E}$ for any $r \geq 3$.

Gelfand-MacPherson Correspondence

We represent n points $p_1, \cdots, p_n$ on $\mathbb{P}^2$ by a matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{12} & \cdots & a_{3n} \end{bmatrix}.$$

The group $\mathrm{GL}_3 \times \mathbb{G}_m^n$ acts on it: GL_3 acts by left multiplication; $\mathbb{G}_m^n$ by multiplying column-wise.

Divide by the action of $\mathbb{G}_m^n$ first, we obtain

$$(\mathbb{P}^2)^n) \text{ with the residual } \mathrm{GL}_3\text{-action.}$$

Divide by the action of GL_3 first, we obtain

$$\mathrm{Gr}^{3,n} \text{ with the residual } \mathbb{G}_m^n\text{-action.}$$

This leads to the Gelfand-MacPherson Correspondence

$$\text{the } \mathrm{GL}_3\text{-orbits on } (\mathbb{P}^2)^n)$$

are in one-to-one correspondence with

$$\mathbb{G}_m^n\text{-orbits on } \mathrm{Gr}^{3,n}.$$

Under the above, we have the correspondence, we have that

$$\mathbf{C}_{\underline{\mathrm{d}}} = \{(\mathbf{p_1}, \cdots, \mathbf{p_n}) \in (\mathbb{P}^\mathbf{2})^\mathbf{n} \mid \textbf{the matroid is } \underline{\mathbf{d}} = (d_I)_{I \subset [n]}\}$$

corresponds to

$$\mathrm{Gr}^{\mathbf{3,n}}_{\underline{\mathrm{d}}} = \{\mathbf{F} \subset \mathbf{k^n} \mid \dim_\mathbf{k} \mathbf{F} \cap \mathbf{E_I} = \mathbf{d_I},\ \forall\, \mathbf{I} \subset [\mathbf{n}]\}$$

where $e_1, \cdots, e_n$ is the basis of $\mathbf{k}^n$ and $E_I = \mathrm{span}\{e_i \mid i \in I\}$.

Lafforgue's version of Mnëv's universality

Theorem. (Mnëv, Lafforgue) Let X be an affine scheme of finite type over $\operatorname{Spec}\mathbb{Z}$. Then, there exists a matroid $\underline{\mathrm{d}}$ of rank 3 on the set $[n]$ such that $(\mathbb{G}_m^n/\mathbb{G}_m)$ acts freely on the matroid Schubert cell $\mathrm{Gr}_{\underline{\mathrm{d}}}^{3,E}$. Further, there exists a positive integer r and an open subset $U \subset X \times \mathbb{A}^r$ projecting onto X such that U is isomorphic to the quotient space $\underline{\mathrm{Gr}}_{\underline{\mathrm{d}}}^{3,E} := \mathrm{Gr}_{\underline{\mathrm{d}}}^{3,E}/(\mathbb{G}_m^n/\mathbb{G}_m)$.

$$\begin{array}{ccccc} \mathrm{Gr}_{\underline{\mathrm{d}}}^{3,E} \longrightarrow \mathrm{Gr}_{\underline{\mathrm{d}}}^{3,E}/(\mathbb{G}_m^n/\mathbb{G}_m) \cong U & \hookrightarrow & X \times \mathbb{A}^r \\ & \searrow & \downarrow \\ & & X \end{array}$$

where the south-east down-arrow is **surjective** so that no singularity is missed.

Local equations of $\mathrm{Gr}^{3,n}_{\underline{\mathbf{d}}}$

The Grassmannian $\mathrm{Gr}^{3,E}$ comes equipped with the (lattice) polytope

$$\Delta^{3,n} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid 0 \le x_\alpha \le 1, \ \forall\, \alpha; \ x_1 + \cdots + x_n = 3\}.$$

For any $\underline{i} = (1 \le i_1 < i_2 < i_3 \le n)$, let $\mathbf{x}_{\underline{i}} = (x_1, \cdots, x_n)$ such that

$$\begin{cases} x_i = 1, & \text{if } i \in \underline{i}, \\ x_i = 0, & \text{otherwise.} \end{cases}$$

Then, it is known that

$$\Delta^{3,n} \cap \mathbb{N}^n = \{\mathbf{x}_{\underline{i}} \mid \underline{i} \in \mathbb{I}_{d,n}\} = \text{the set of the vertices of } \Delta^{3,n}.$$

A matroid $\underline{\mathrm{d}} = (d_I)_{I \subset [n]}$ defines the subpolytope of $\Delta^{3,n}$

$$\Delta^{3,n}_{\underline{\mathrm{d}}} = \{(x_1, \cdots, x_n) \in \Delta^{3,n} \mid \sum_{\alpha \in I} x_\alpha \ge d_I, \ \forall\, I \subset [n]\}.$$

Theorem. (Lafforgue) Let $\underline{\mathrm{d}}$ be any matroid of rank d on the set $[n]$ as considered above. Then, in the Grassmannian

$$\mathrm{Gr}^{3,n} \hookrightarrow \mathbb{P}(\wedge^3 \mathbf{k}^n) = \{(p_{\underline{i}}) \in \mathbb{G}_m \backslash (\wedge^3 \mathbf{k}^n \backslash \{0\})\},$$

$\mathrm{Gr}^{3,n}_{\underline{\mathrm{d}}}$, as a locally closed subscheme, is defined by

$$p_{\underline{i}} = 0, \quad \forall\, \mathbf{x}_{\underline{i}} \notin \Delta^{3,n}_{\underline{\mathrm{d}}},$$

$$p_{\underline{i}} \neq 0, \quad \forall\, \mathbf{x}_{\underline{i}} \in \Delta^{3,n}_{\underline{\mathrm{d}}}.$$

$\mathrm{Gr}^{3,n}_{\underline{\mathrm{d}}}$ is cut-out by the coordinate hyperplanes $p_{\underline{i}} = 0, \mathbf{x}_{\underline{i}} \notin \Delta^{d,n}_{\underline{\mathrm{d}}}$

Some heuristic discussions

• Each divisor $\mathrm{Gr}^{3,n}_{\underline{\mathrm{d}}} \cap (p_{\underline{i}} = 0)$ is reasonable, but their intersections can be arbitrary.

• Goal: Blow up these divisors to make them intersecting transversally.

• What to blow up, and how?

• To find clues, let's look at Plücker relations, they are in the forms

$$\bar{F}_{(123),1uv} = x_{1uv} - x_{12u}x_{13v} + x_{13u}x_{12v},$$
$$\bar{F}_{(123),2uv} = x_{2uv} - x_{12u}x_{23v} + x_{23u}x_{12v},$$
$$\bar{F}_{(123),3uv} = x_{3uv} - x_{13u}x_{23v} + x_{23u}x_{13v},$$
$$\bar{F}_{(123),abc} = x_{abc} - x_{12a}x_{3bc} + x_{13a}x_{2bc} - x_{23a}x_{1bc},$$

(These relations are for the chart $U = (p_{123} = 1)$. By permutation of indexes, any $\mathrm{Gr}^{3,n}_{\underline{\mathrm{d}}}$ is contained in the chart $U = (p_{123} = 1)$.)

Blowing up the intersection of two divisors, e.g.,

$$x_{12u} = x_{13u} = 0, \text{ or, } x_{12u} = x_{12v} = 0, \text{ etc.}$$

can make Plücker relations so messy that we quickly lose control of their forms, let alone analyse their final forms (by experience).

• My intuition, or put it in a nicer term, my insights are: if I can separate the terms of Plücker relations into binomials, then it is likely to reach the goal.

New models of $\mathrm{Gr}^{3,n}_{\underline{\mathbf{d}}}$ and "better" local equations

- Also, keep in mind that we want to blow up the loci like

$$x_{12u} = x_{13u} = 0, \text{ or, } x_{12u} = x_{12v} = 0, \text{ etc.}$$

and these loci also make the terms of Plücker relations vanish.

So, as the very first step, we let $\mathcal{V}$ be the closure of U in

$$U \times \prod_F \mathbb{P}_F$$

where F runs over the Plücker relations in the previous page, and for

$$\bar{F}_{(123),1uv} = x_{1uv} - x_{12u}x_{13v} + x_{13u}x_{12v}$$

$$\mathbb{P}_F = \{[x_{(123,1uv)}, x_{(12u,13v)}, x_{(13u,12v)}]\} \cong \mathbb{P}^2$$

the same for $\bar{F}_{(123),2uv}$ and $\bar{F}_{(123),3uv}$; for

$$\bar{F}_{(123),abc} = x_{abc} - x_{12a}x_{3bc} + x_{13a}x_{2bc} - x_{23a}x_{1bc},$$

$$\mathbb{P}_F = \{[x_{(123,abc)}, x_{(12a,3bc)}, x_{(13a,2bc)}, x_{(12a,3bc)}, x_{(23a,1bc)}]\} \cong \mathbb{P}^3.$$

If we write every Plücker relation as

$$F : \sum_{s \in S_F} \mathrm{sgn}(s) x_{\underline{u}_s} x_{\underline{v}_s},$$

then we have a rational map

$$U \dashrightarrow \prod_F \mathbb{P}_F$$

$$p \to \prod_F [x_{\underline{u}_s} x_{\underline{v}_s}]_{s \in S_F},$$

e.g.,

$$[x_{1uv}, x_{12u}x_{13v}, x_{13u}x_{12v}].$$

Our new birational model of the chart U

- $\mathcal{V}$ is the graph of this rational map, and is birational to U.

What do we gain? **Better and neat relations to handle!**

$$x_{1uv}x_{(12u,13v)} - x_{12u}x_{13v}x_{(123,1uv)},\ x_{1uv}x_{(13u,12v)} - x_{13u}x_{12v}x_{(123,1uv)},$$

$$x_{2uv}x_{(12u,23v)} - x_{12u}x_{23v}x_{(123,2uv)},\ x_{2uv}x_{(23u,12v)} - x_{23u}x_{12v}x_{(123,2uv)},$$

$$x_{3uv}x_{(13u,23v)} - x_{13u}x_{23v}x_{(123,3uv)},\ x_{3uv}x_{(23u,13v)} - x_{23u}x_{12v}x_{(123,3uv)},$$

$$x_{abc}x_{(12a,3bc)} - x_{12a}x_{3bc}x_{(123,abc)},\ x_{abc}x_{(13a,2bc)} - x_{13a}x_{2bc}x_{(123,abc)},$$

$$x_{abc}x_{(23a,1bc)} - x_{23a}x_{1bc}x_{(123,abc)}.$$

We see that the terms of all the $\underline{m}$-primary Plücker relations are separated into the two terms of the above binomials.

And, the Plücker relations become linear relations

$$L_{(123),1uv} = x_{(123,1uv)} - x_{(12u,13v)} + x_{(13u,12v)},$$

$$L_{(123),2uv} = x_{(123,2uv)} - x_{(12u,23v)} + x_{(23u,12v)},$$

$$L_{(123),3uv} = x_{(123,3uv)} - x_{(13u,23v)} + x_{(23u,13v)},$$

$$L_{(123),abc} = x_{(123,abc)} - x_{(}(12a,3bc) + x_{(13a,2bc)} - x_{(23a,1bc)}.$$

These are the governing relations:

- they guide me the blowing up process, and in the end,
- they remain as the only defining relations,
- all the others eventually become dependent!

Here are the other relations

$$x_{1bc}x_{2b'c'}x_{(13a,2bc)}x_{(23a,1b'c')} - x_{2bc}x_{1b'c'}x_{(23a,1bc)}x_{(13a,2b'c')}$$

$$x_{12b}x_{3ac}x_{(13b,2\bar{b}\bar{c})}x_{(13\bar{a},2ac)}x_{(12\bar{a},3\bar{b}\bar{c})} - x_{13b}x_{2ac}x_{(12b,3\bar{b}\bar{c})}x_{(12\bar{a},3ac)}x_{(13\bar{a},2\bar{b}\bar{c})}$$

$$x_{13b}x_{2ac}x_{(12b,13a')}x_{(12a',3\bar{b}\bar{c})}x_{(12\bar{a},3ac)}x_{(13\bar{a},2\bar{b}\bar{c})} - x_{12b}x_{3ac}x_{(13b,12a')}x_{(13a',2\bar{b}\bar{c})}x_{(13\bar{a},2ac)}x_{(12\bar{a},3\bar{b}\bar{c})}$$

$$x_{(12a,13b)}x_{(13a,12c)}x_{(12b,13c)} - x_{(13a,12b)}x_{(12a,13c)}x_{(13b,12c)}.$$

$$x_{(12a,13b)}x_{(13a,12c)}x_{(12b,23c)}x_{(23b,13c)} - x_{(13a,12b)}x_{(12a,13c)}x_{(23b,12c)}x_{(13b,23c)}.$$

$$x_{(12a,3bc)}x_{(13a,2\bar{b}\bar{c})}x_{(13\bar{a},2bc)}x_{(12\bar{a},3\bar{b}\bar{c})} - x_{(13a,2bc)}x_{(12a,3\bar{b}\bar{c})}x_{(12\bar{a},3bc)}x_{(13\bar{a},2\bar{b}\bar{c})}.$$

$$x_{(12a,13a')}x_{(13a,2bc)}x_{(12a',3\bar{b}\bar{c})}x_{(12\bar{a},3bc)}x_{(13\bar{a},2\bar{b}\bar{c})} - x_{(13a,12a')}x_{(12a,3bc)}x_{(13a',2\bar{b}\bar{c})}x_{(13\bar{a},2bc)}x_{(12\bar{a},3\bar{b}\bar{c})}$$

These will become dependent and can be discarded in the end:

Meanwhile, they possess the following useful properties

- they are square-free.
- they are linear in variables of $\mathbb{P}_F$ for every Plücker relation F.

The process to reach the goal

- The blowups are performed on blocks of equations, block by block.

$$x_{1uv}x_{(12u,13v)} - x_{12u}x_{13v}x_{(123,1uv)},$$

$$x_{1uv}x_{(13u,12v)} - x_{13u}x_{12v}x_{(123,1uv)},$$

$$L_{(123),1uv} = x_{(123,1uv)} - x_{(12u,13v)} + x_{(13u,12v)}.$$

$$x_{2uv}x_{(12u,23v)} - x_{12u}x_{23v}x_{(123,2uv)},$$

$$x_{2uv}x_{(23u,12v)} - x_{23u}x_{12v}x_{(123,2uv)},$$

$$L_{(123),2uv} = x_{(123,2uv)} - x_{(12u,23v)} + x_{(23u,12v)},$$

$$x_{3uv}x_{(13u,23v)} - x_{13u}x_{23v}x_{(123,3uv)},$$

$$x_{3uv}x_{(23u,13v)} - x_{23u}x_{12v}x_{(123,3uv)},$$

$$L_{(123),3uv} = x_{(123,3uv)} - x_{(13u,23v)} + x_{(23u,13v)},$$

$$x_{abc}x_{(12a,3bc)} - x_{12a}x_{3bc}x_{(123,abc)},$$

$$x_{abc}x_{(13a,2bc)} - x_{13a}x_{2bc}x_{(123,abc)},$$

$$x_{abc}x_{(23a,1bc)} - x_{23a}x_{1bc}x_{(123,abc)}.$$

$$L_{(123),abc} = x_{(123,abc)} - x_{(12a,3bc)} + x_{(13a,2bc)} - x_{(23a,1bc)}.$$

Blowup centers are

$$\mathcal{Z}_\vartheta : (x_{1uv} = 0) \cap (x_{(123,1uv)} = 0), \quad (x_{abc} = 0) \cap (x_{(123,abc)} = 0).$$

$\mathcal{Z}_\wp : (x_{1uv} = 0) \cap (x_{12u} = 0), \quad (x_{abc} = 0) \cap (x_{13a} = 0),$ and, much more

$$\mathcal{Z}_\ell : (L_{(123),3uv} = 0) \cap (x_{(123,3uv)} = 0).$$

How do we know we achieve our goal

The blowing up process is complex, many details are subtle......

In the end, by a 10-page calculation, we obtain

$$J^* = \begin{pmatrix} J^*(\mathfrak{G}_{\mathfrak{V},F_1}|_{\widetilde{\Gamma}_{\mathfrak{V}}}) & 0 & 0 & \cdots & 0 \\ * & J^*(\mathfrak{G}_{\mathfrak{V},F_2}|_{\widetilde{\Gamma}_{\mathfrak{V}}}) & 0 & \cdots & 0 \\ \vdots & & & & \\ * & * & * & \cdots & J^*(\mathfrak{G}_{\mathfrak{V},F_\Upsilon}|_{\widetilde{\Gamma}_{\mathfrak{V}}}) \end{pmatrix}. \tag{0.1}$$

such that all the blocks along diagonal are invertible.

This implies

Theorem. For the birational model $\mathcal{V}$ of the chart $U \subset \mathrm{Gr}^{3,n}$, there exists a resolution $\widetilde{\mathcal{V}} \longrightarrow \mathcal{V}$ such that the boundary of $\widetilde{\mathcal{V}}$ is a simple normal crossing divisor.

Then, by finding birational slices (hard), we obtain

Theorem. For any matroid, the stratum $\mathrm{Gr}^{3,n}_{\underline{\mathrm{d}}}$ admits a resolution

$$\widetilde{\mathrm{Gr}}^{3,n}_{\underline{\mathrm{d}}} \longrightarrow \mathrm{Gr}^{3,n}_{\underline{\mathrm{d}}}.$$

I am rewriting to replace the paper on arxiv:

- focusing on $\mathrm{Gr}^{3,n}$ only, so will be more explicit.
- the arxiv paper contains errors, will update once and for all when a new version on $\mathrm{Gr}^{3,n}$ is completed.
- your comments are always welcome, just email me!

THANK YOU!