Relative mirror symmetry and the proper Landau–Ginzburg potential

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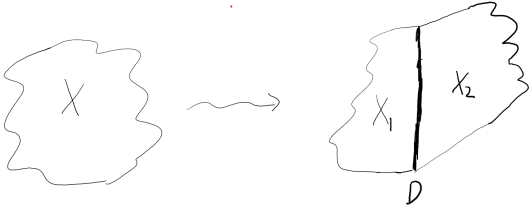
A big picture

A-model $\underset{\longleftarrow}{\overset{\text{Mirror Symmetry}}{\longrightarrow}}$ B-model

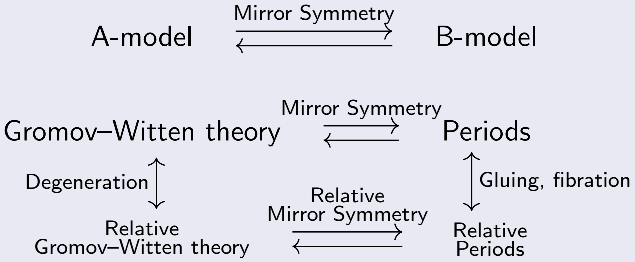
Symplectic Geometry of X $\underset{\longleftarrow}{\overset{\text{Mirror Symmetry}}{\longrightarrow}}$ Complex geometry of $X^\vee$

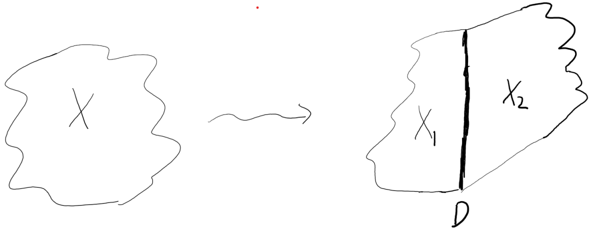
Degeneration $\updownarrow$ | $\updownarrow$ Fibration

Symplectic geometry of $(X_1,D)(X_2,D)$ $\underset{\longleftarrow}{\overset{\text{Relative Mirror Symmetry}}{\longrightarrow}}$ Complex geometry of $(X_1,D)^\vee$ and $(X_2,D)^\vee$



Enumerative mirror symmetry





Absolute Gromov–Witten invariants

Moduli Space of Stable Maps

$\overline{M}_{g,n}(X,d)$: the moduli space of stable maps of degree d from genus g nodal curves with n-markings to a smooth projective variety X. It consists of

$$(C,\{p_i\}_{i=1}^n)\xrightarrow{f} X,$$

where

- C is a projective, connected, nodal curve of genus g;
- $p_1,\ldots,p_n$ are distinct non-singular points of C;
- $f_*[C]=d\in H_2(X)$;
- stable: automorphisms of the maps are finite

Definition

- For each marking p_i, there is an evaluation map:

$$\mathrm{ev}_i : \overline{M}_{g,n}(X,d) \to X$$
$$\{(C, \{p_i\}_{i=1}^n) \xrightarrow{f} X\} \mapsto f(p_i).$$

- For each marking p_i, there is a tautological line bundle L_i over $\overline{M}_{g,n}(X,d)$ whose fiber is the cotangent space of the curve at p_i. Let

$$\psi_i = c_1(L_i).$$

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Definition

Given cohomological classes $\gamma_i \in H^*(X)$, and $a_i \in \mathbb{Z}_{\geq 0}$, one can define the Gromov–Witten invariant

$$\left\langle \prod_{i=1}^n \tau_{a_i}(\gamma_i) \right\rangle_{g,n,d}^X := \int_{[\overline{M}_{g,n}(X,d)]^{\mathrm{vir}}} \prod_{i=1}^n (\mathrm{ev}_i^* \gamma_i) \psi_i^{a_i}.$$

Enumerative mirror symmetry

- The A-model data is a generating function of genus zero Gromov–Witten invariants called the J-function $J_X(\tau, z)$:

$$J_X(\tau, z) = z + \tau + \sum_{\substack{(\beta,l)\neq(0,0),(0,1)\\ \beta\in \mathrm{NE(X)}}} \sum_\alpha \frac{q^\beta}{l!} \left\langle \frac{\phi_\alpha}{z-\psi}, \tau, \ldots, \tau \right\rangle^X_{0,1+l,\beta} \phi^\alpha,$$

  where $\tau = \tau_{0,2} + \tau' \in H^*(X)$; $\tau_{0,2} = \sum_{i=1}^r p_i \log q_i \in H^2(X)$; $\tau' \in H^*(X) \smallsetminus H^2(X)$; NE(X) is the cone of effective curve classes in X; $\{\phi_\alpha\}$ is a basis of $H^*(X)$; $\{\phi^\alpha\}$ is the dual basis under the Poincaré pairing.
- The B-model data is period integrals called the I-function $I_X(y, z)$

Mirror theorem (Givental 1996, Lian–Liu–Yau 1997, ...)

$$J_X(\tau(y), z) = I_X(y, z)$$

where $\tau(y)$ is called the mirror map.

I-function

For quintic threefold the I-function is

$$I_X(y) = \sum_{d\geq 0} y^{H+d} \frac{\prod_{a=1}^{5d}(5H+a)}{\prod_{a=1}^{d}(H+a)^5}$$
$$= \sum_{d\geq 0} \frac{(5d)!}{(d!)^5} y^d H^0 + O(H).$$

Remark

Such a mirror theorem has been proved in many cases including toric stacks, Grassmannians, partial flag varieties etc.

Relative Gromov–Witten invariants

Relative Gromov–Witten theory is the enumerative theory of counting curves with tangency condition along a divisor (a codimensional one subvariety).

- X: a smooth projective variety.
- D: a smooth divisor of X.
- For $d \in H_2(X, \mathbb{Q})$, we consider a partition $\vec{k} = (k_1, \ldots, k_m)$ of $\int_d [D]$. That is,

$$\sum_{i=1}^{m} k_i = \int_d [D], \quad k_i > 0$$

- $\overline{M}_{g,\vec{k},n,d}(X, D)$: the moduli space of $(m+n)$-pointed, genus g, degree $d \in H_2(X, \mathbb{Q})$, relative stable maps to (X, D) such that the relative conditions are given by the partition $\vec{k}$.

Relative Gromov–Witten invariants

Evaluation Maps

There are two types of evaluation maps.

$$\mathrm{ev}_i : \overline{M}_{g,\vec{k},n,d}(X,D) \to D, \quad \text{for } 1 \leq i \leq m;$$
$$\mathrm{ev}_i : \overline{M}_{g,\vec{k},n,d}(X,D) \to X, \quad \text{for } m+1 \leq i \leq m+n.$$

The first m markings are relative markings with contact order k_i, the last n markings are interior markings.

Data

- $\delta_i \in H^*(D,\mathbb{Q})$, for $1 \leq i \leq m$.
- $\gamma_{m+i} \in H^*(X,\mathbb{Q})$, for $1 \leq i \leq n$.

Definition

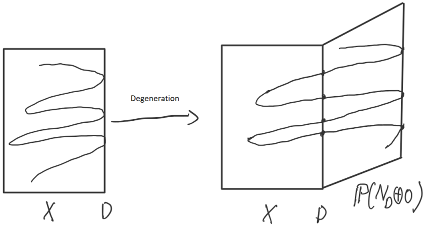
Definition

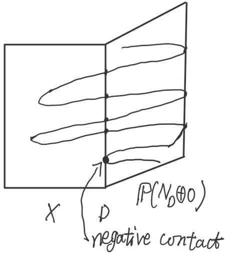
Relative Gromov–Witten invariants of (X, D) *are defined as*

$$\left\langle \prod_{i=1}^{m} \tau_{a_i}(\delta_i) \middle| \prod_{i=1}^{n} \tau_{a_i}(\gamma_{m+i}) \right\rangle_{g,\vec{k},n,d}^{(X,D)} := \tag{1}$$

$$\int_{[\overline{M}_{g,\vec{k},n,d}(X,D)]^{vir}} \prod_{i=1}^{m} \mathrm{ev}_i^*(\delta_i)\bar{\psi}_i^{a_i} \prod_{i=1}^{n} \mathrm{ev}_{m+i}^*(\gamma_{m+i})\bar{\psi}_i^{a_i}.$$

Relative invariants with negative contact orders (Fan–Wu–Y, 2020)





Structure of relative Gromov–Witten theory (Fan–Wu–Y)

Relative Gromov–Witten invariants with (possibly) negative contact orders have the following properties:

- Relative quantum cohomology ring
- Topological recursion relation
- WDVV equation
- Givental's formalism: Givental's symplectic vector space, Lagrangian cone etc.
- Virasoro constraint (in genus zero).

While relative Gromov–Witten theory without negative contact orders does not satisfy these properties.

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Remark

For those who are familiar with Gross–Siebert program, similar invariants also appear in their program where they are called punctured Gromov–Witten invariants.

Relative mirror theorem

We have the following Givental-style mirror theorem.

Theorem (Fan–Tseng–Y, 2019)

Assume that D is nef, the I-function for the pair (X, D) is

$$I_{(X,D)}(t,z) = \sum_{d \in \overline{\mathrm{NE}}(X)} J_{X,d}(t,z) y^d \left(\prod_{0 < a \leq D\cdot d - 1} (D + az) \right) [\mathbf{1}]_{-D\cdot d}.$$

When $-K_X - D$ is nef, we have

$$J_{(X,D)}(\tau(t), z) = I_{(X,D)}(t,z),$$

where the change of variable $\tau(t)$ is called the relative mirror map.

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Question

What is the mirror of a pair?

What is the mirror of a smooth pair?

Recall,

Mirror symmetry for Fano toric varieties

Following Givental, the mirror of an n-dimensional Fano toric variety X is a Landau–Ginzburg model $((\mathbb{C}^\times)^n, f)$, where $f : (\mathbb{C}^\times)^n \to \mathbb{C}$ is called the super-potential.

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In general, it is expected the following

Mirror symmetry for Fano varieties

The mirror of a Fano variety X is a Landau–Ginzburg model $(X^\vee, W)$ consisting of a Kähler manifold $X^\vee$ satisfying $h^1(X^\vee) = 0$ and a proper map $W : X^\vee \to \mathbb{C}$, where W is called the superpotential.
Furthermore, the generic fiber of W should be mirror to the smooth anticanonical divisor of X.

Therefore, a proper LG model $(X^\vee, W)$ is actually mirror to a smooth log Calabi–Yau pair (X, D).
Givental's LG model is actually mirror to a toric variety with its toric boundary.

Mirror symmetry for log Calabi–Yau pairs

For a smooth log Calabi–Yau pair (X, D)

$$\text{Log Calabi–Yau } (X, D) \quad \underset{}{\overset{\text{Mirror}}{\rightleftarrows}} \quad \text{LG model } (X^\vee, W)$$

$$\text{Noncompact Calabi–Yau } X \setminus D \quad \overset{\text{Mirror}}{\rightleftarrows} \quad X^\vee \text{ (without } W)$$

$$\text{Smooth anticanonical divisor } D \quad \overset{\text{Mirror}}{\rightleftarrows} \quad W^{-1}(t) \text{ for generic } t \in \mathbb{C}$$

Intrinsic mirror symmetry

For relative mirror symmetry, we need the construction of the variety $X^\vee$ and the potential W.

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A general construction of the variety $X^\vee$ is through intrinsic mirror symmetry in the Gross–Siebert program. One considers a maximally unipotent degeneration $g : Y \to S$, where S is an affine curve, of the pair (X, D).

Intrinsic mirror symmetry

For relative mirror symmetry, we need the construction of the variety $X^\vee$ and the potential W.

Intrinsic mirror construction

The variety $X^\vee$ is the mirror of the complement $X \setminus D$.
A general construction of the variety $X^\vee$ is through intrinsic mirror symmetry in the Gross–Siebert program. One considers a maximally unipotent degeneration $g : Y \to S$, where S is an affine curve, of the pair (X, D).
The mirror $X^\vee$: the projective spectrum of the degree zero part of the relative quantum cohomology $QH^0_{\log}(Y, D')$ of (Y, D'), where D' is a certain divisor that contains $g^{-1}(0)$.

The proper LG potential from instrinsic mirror symmetry

The LG potential is also constructed from the instrinsic mirror symmetry.

Theta function

Let $QH^0_{\log}(X, D)$ be the degree zero subalgebra of the relative quantum cohomology ring $\mathrm{QH}^*_{\log}(X, D)$ of a smooth log Calabi–Yau pair (X, D). The set

$$\{\vartheta_p\}, p \in \mathbb{Z}_{\geq 0}$$

of theta functions form a canonical basis of $\mathrm{QH}^0_{\log}(X, D)$.

The construction

The base of the Landau–Ginzburg mirror of (X, D) is $\operatorname{Spec} QH^0_{\log}(X, D) = \mathbb{A}^1$ and the superpotential is $W = \vartheta_1$, the unique primitive theta function of $QH^0_{\log}(X, D)$.

Theta function

What are theta functions?

Theta functions satisfy the following multiplication rule

$$\vartheta_{p_1} \star \vartheta_{p_2} = \sum_{r \geq 0, \beta} N^{\beta}_{p_1, p_2, -r} \vartheta_r. \tag{2}$$

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Structure constants

The structure constants $N^{\beta}_{p_1, p_2, -r}$ are defined as the invariants of (X, D) with two "inputs" with positive contact orders given by $p_1, p_2 \in B(\mathbb{Z})$, one "output" with negative contact order given by $-r$ such that $r \in B(\mathbb{Z})$, and a point constraint for the punctured point. Namely,

$$N^{\beta}_{p_1, p_2, -r} = \langle [1]_{p_1}, [1]_{p_2}, [\mathrm{pt}]_{-r} \rangle^{(X,D)}_{0,3,\beta}. \qquad (3)$$

Structure constants

We have the following identity for structure constants.

Proposition (Y, 2022 & Yu Wang, 2022)

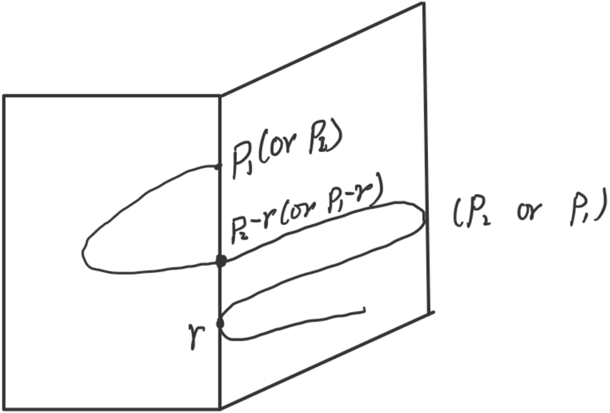
The structure constants $N^{\beta}_{p_1,p_2,-r}$ can be written as two-point relative invariants (without negative contact):

$$N^{\beta}_{p_1,p_2,-r} = (p_1 - r)\langle [\mathrm{pt}]_{p_1-r}, [1]_{p_2}\rangle^{(X,D)}_{0,2,\beta} + (p_2 - r)\langle [\mathrm{pt}]_{p_2-r}, [1]_{p_1}\rangle^{(X,D)}_{0,2,\beta}$$

Proof of the proposition

$$N^{\beta}_{p_1,p_2,-r} = (p_1 - r)\langle[\mathrm{pt}]_{p_1-r}, [1]_{p_2}\rangle^{(X,D)}_{0,2,\beta} + (p_2 - r)\langle[\mathrm{pt}]_{p_2-r}, [1]_{p_1}\rangle^{(X,D)}_{0,2,\beta}$$

Proof by picture:



Theta functions

We define the theta function in terms of two-point relative Gromov–Witten invariants.

Definition

For $p \geq 1$, the theta function is

$$\vartheta_p := x^{-p} + \sum_{n=1}^{\infty} nN_{n,p} t^{n+p} x^n, \tag{4}$$

where

$$N_{n,p} = \sum_{\beta} \langle [\mathrm{pt}]_n, [1]_p \rangle_{0,2,\beta}^{(X,D)}.$$

Theta functions

Proposition (Y, 2022)

The above definition of the theta functions satisfy the multiplication rule

$$\vartheta_{p_1} \star \vartheta_{p_2} = \sum_{r \geq 0, \beta} N^{\beta}_{p_1, p_2, -r} \vartheta_r. \tag{5}$$

Proof.

By the WDVV equation for relative Gromov–Witten theory and the above identity for structure constants. □

The work of Grafnitz–Ruddat–Zaslow, 2022

Conjecture & Title of the paper by [GRZ]:

"The proper Landau–Ginzburg potential is the open mirror map."

Theorem (GRZ)

For a toric del Pezzo surface X with a smooth anticanonical divisor D, the proper potential is

$$W = \textit{open mirror map of } K_X = \sum n_0^{\text{open}}(K_X),$$

a generating function of genus zero open Gromov–Witten invariants of K_X.

The work of Grafnitz–Ruddat–Zaslow

Proof:

Step 1 Use tropical definition of the LG potential in Carl–Pumperla–Siebert to define the potential in terms of tropical disk countings. Then relate tropical disks to tropical curves.

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Step 2 Tropical to log invariants

$$N^{\mathrm{trop}}_{p,q}(X,D,\beta) = pN^{\log}_{p,q}(X,D,\beta)$$

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Step 3 Theta calculations

$$N^{\log}_{1,n}(X,D,\beta) = n^2 N^{\log}_{n,1}(X,D,\beta)$$

The work of Grafnitz–Ruddat–Zaslow

Step 4 Trading a contact point for a blow-up.

$$N_n(\tilde{X}, \tilde{D}, \pi^{\beta} - C) = N_{1,n}(X, D, \beta)$$

The work of Grafnitz–Ruddat–Zaslow

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Step 5 Log-local principle:

$$N_n(\tilde{X}, \tilde{D}, \pi^{\beta} - C) = (-1)^{n+1} n \cdot N(K_{\tilde{X}}, \beta)$$

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$$N_n(\tilde{X}, \tilde{D}, \pi^\beta - C) = (-1)^{n+1} n \cdot N(K_{\tilde{X}}, \beta)$$

Step 6 Closed-open correspondence of Lau–Leung–Wu:

$$N_n(\tilde{X}, \tilde{D}, \pi^\beta - C) = n^{\text{open}}_{\beta+\beta_0}(K_X).$$

Relative mirror theorem

We will compute the proper potential $W = \vartheta_1$ in general using the relative mirror theorem of [Fan–Tseng–Y, 2019]. Recall,

Theorem (Fan–Tseng–Y, 2019)

Assume that D is nef, the I-function for the pair (X, D) is

$$I_{(X,D)}(t,z) = \sum_{d\in\overline{\mathrm{NE}}(X)} J_{X,d}(t,z)y^d \left(\prod_{0<a\leq D\cdot d-1}(D+az)\right)[\mathbf{1}]_{-D\cdot d}.$$

When $-K_X - D$ is nef, we have

$$J_{(X,D)}(\tau(t),z) = I_{(X,D)}(t,z),$$

where the change of variable $\tau(t)$ is called the relative mirror map.

Relative mirror map

The relative mirror map is the z^0-coefficient of the relative I-function.

Relative mirror map

$$\tau(y) = \sum_{i=1}^{r} p_i \log y_i + \sum_{\substack{\beta \in \mathrm{NE}(X) \\ D\cdot\beta \geq 2}} \langle [\mathrm{pt}]\psi^{D\cdot\beta-2}\rangle^X_{0,1,\beta} y^\beta (D\cdot\beta-1)![1]_{-D\cdot\beta}.$$

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Remark

However, this I-function only computes one-point invariants. To compute two-point invariants, we need the extended I-function.

The extended I-function

Question

What is the extended I-function.

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Recall

relative vs orbifold invariants

[Abramovich–Cadman–Wise, 2017] and [Fan–Wu–Y, 2020],: genus zero relative invariants equal to genus zero orbifold invariants of root stacks $X_{D,r}$ for $r \gg 1$.

$$GW_0(X, D) = GW_0(X_{D,r}), \text{ for } r \gg 1$$

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$$GW_0(X,D) = GW_0(X_{D,r}), \text{ for } r \gg 1$$

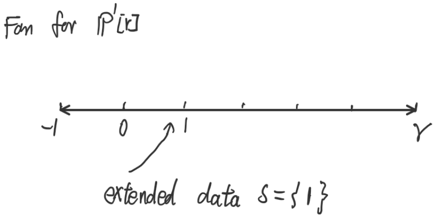
A hypersurface construction of root stacks

[Fan–Tseng–Y]: the root stack $X_{D,r}$ is a hypersurface of a $\mathbb{P}^1[r]$-bundle over X.

The extended I-function

The mirror theorem for toric stack bundles

[Jiang–Tseng–Y, 2017]: S-extended I-function for toric stack bundles.



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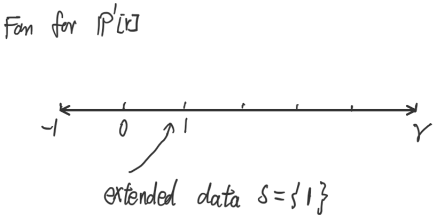

Remark

To compute ϑ_1, we just need to consider the simplest case of the extended I-function where the extended data is

$$S = \{1\}.$$

The extended I-function

Definition

The S-extended I-function of (X, D) is defined as follows.

$$I^S_{(X,D)}(y, x_1, z) = I_+ + I_-,$$

where

$$I_+ := \sum_{\substack{\beta \in \mathrm{NE}(X), k \in \mathbb{Z}_{\geq 0} \\ k < D\cdot\beta}} J_{X,\beta}(\tau_{0,2}, z) y^\beta \frac{x_1^k}{z^k k!} \frac{\prod_{0<a\leq D\cdot\beta}(D + az)}{D + (D\cdot\beta - k)z} [1]_{-D\cdot\beta+k},$$

and

$$I_- := \sum_{\substack{\beta \in \mathrm{NE}(X), k \in \mathbb{Z}_{\geq 0} \\ k \geq D\cdot\beta}} J_{X,\beta}(\tau_{0,2}, z) y^\beta \frac{x_1^k}{z^k k!} \left(\prod_{0<a\leq D\cdot\beta} (D + az) \right) [1]_{-D\cdot\beta+k}.$$

The extended I-function

Recall that the J-function is

$$J_{(X,D)}(\tau, z) = z + \tau + \sum_{\substack{(\beta,l)\neq(0,0),(0,1)\\ \beta\in \mathrm{NE}(X)}} \sum_{\alpha} \frac{q^\beta}{l!} \left\langle \frac{\phi_\alpha}{z-\bar{\psi}}, \tau, \ldots, \tau \right\rangle_{0,1+l,\beta}^{(X,D)} \phi^\alpha,$$

Theorem (Fan–Tseng–Y, 2019)

Let X be a smooth projective variety and D be a smooth nef divisor such that $-K_X - D$ is nef. Then

$$J(\tau(y,x_1),z) = I(y,x_1,z).$$

via change of variables, called the extended relative mirror map $\tau(y,x_1) =$

$$\sum_{i=1}^{r} p_i \log y_i + x_1[1]_1 + \sum_{\substack{\beta\in\mathrm{NE}(X)\\ D\cdot\beta\geq 2}} \langle [\mathrm{pt}]\psi^{D\cdot\beta-2}\rangle_{0,1,\beta}^{X} y^\beta (D\cdot\beta-1)![1]_{-D\cdot\beta}. \quad (6)$$

Computation

We need to compute the following invariants.

- Relative invariants with one positive contact order and several negative contact orders:

$$\langle [1]_{-k_1}, \cdots, [1]_{-k_l}, [\gamma]_{k_{l+1}} \bar{\psi}^a \rangle_{0,l+1,\beta}^{(X,D)}, k_i > 0;$$

- Relative invariants with two positive contact orders and several negative contact orders of the following form:

$$\langle [1]_1, [1]_{-k_1}, \cdots, [1]_{-k_l}, [\gamma]_{k_{l+1}} \rangle_{0,l+2,\beta}^{(X,D)}, k_i > 0;$$

- Degree zero relative invariants with two positive contact orders and several negative contact orders of the following form:

$$\langle [1]_1, [1]_{-k_1}, \cdots, [1]_{-k_l}, [\mathrm{pt}]_{k_{l+1}} \rangle_{0,l+2,0}^{(X,D)}; k_i > 0.$$

The proper LG potential is the relative mirror map

After extracting certain coefficients of the J-function and the I-function, take derivatives, and match coefficients, we have

Theorem (Y, 2022)

Let X be a smooth projective variety with a smooth nef anticanonical divisor D. Let $W := \vartheta_1$ be the mirror proper Landau–Ginzburg potential. Set $q^\beta = t^{D\cdot\beta}x^{D\cdot\beta}$. Then

$$W = x^{-1}\exp\left(g(y(q))\right),$$

where

$$g(y) = \sum_{\substack{\beta\in \mathrm{NE}(X)\\ D\cdot\beta\geq 2}} \langle [\mathrm{pt}]\psi^{D\cdot\beta-2}\rangle^X_{0,1,\beta}\, y^\beta (D\cdot\beta-1)!$$

and $y = y(q)$ is the inverse of the relative mirror map

$$\sum_{i=1}^r p_i \log q_i = \sum_{i=1}^r p_i \log y_i + g(y)D. \tag{7}$$

The proper LG potential is the relative mirror map

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This is a natural expectation from the point of view of relative mirror symmetry. Recall that the proper Landau–Ginzburg model $(X^\vee, W)$ is mirror to the smooth log Calabi–Yau pair (X, D). The relative mirror theorem [Fan–Tseng–Y] relates relative Gromov–Witten invariants with relative periods (relative I-functions) via the relative mirror map.

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In order to have a mirror construction with a trivial mirror map, the proper Landau–Ginzburg potential W should encode the instanton corrections and the instanton corrections should be the inverse relative mirror map.

Open mirror map

relative and open mirror maps

The open invariants of $\mathcal{O}_X(-D)$ encoding the instanton corrections are expected to be the inverse mirror map of the local Gromov–Witten theory of $\mathcal{O}_X(-D)$. We observe that the relative mirror map and the local mirror map coincide up to a sign. Therefore, the works of Chan, Cho, Lau, Leung, Tseng on open Gromov–Witten invariants of toric Calabi–Yaus imply the following theorem.

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Theorem (Y, 2022)

Let (X, D) be a smooth log Calabi–Yau pair, such that X is toric and D is nef. The proper Landau–Ginzburg potential of (X, D) is the open mirror map of the local Calabi–Yau manifold $\mathcal{O}_X(-D)$.

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Corollary

The open-closed duality implies the proper Landau–Ginzburg potential is the open mirror map.

SYZ mirror symmetry

Remark

We consider these results provide a complete story from algebro-geometric point of view. It provides a connection between the Gross–Siebert mirror construction and the relative version of the enumerative mirror symmetry (Givental-style mirror symmetry) of Fan–Tseng–Y.

For the SYZ mirror symmetry, one may naturally expect that the proper potential is a generating function of genus zero open Gromov–Witten invariants of $X \setminus D$.

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For the SYZ mirror symmetry, one may naturally expect that the proper potential is a generating function of genus zero open Gromov–Witten invariants of $X \setminus D$.

Remark

Open Gromov–Witten invariants of $\mathcal{O}_X(-D)$ appear here maybe because there exists an identity between open Gromov–Witten invariants of $\mathcal{O}_X(-D)$ and the open Gromov–Witten invariants of $X \setminus D$, similar to the local-relative correspondence for closed Gromov–Witten invariants. However, we do not know if such an identity will be true in general.

Connection to the Fanosearch program

For Fano varieties, we observed that the function

$$g(y) = \sum_{\substack{\beta \in \mathrm{NE}(X) \\ D\cdot\beta \geq 2}} \langle [\mathrm{pt}]\psi^{D\cdot\beta-2}\rangle^X_{0,1,\beta} y^\beta (D\cdot\beta - 1)!$$

is closely related to the regularized quantum periods in the Fano search program.

Theorem (Y, 2022)

The function $g(y)$ coincides with the anti-derivative of the regularized quantum period.

Connection to the Fanosearch program

Remark

It is expected that regularized quantum periods of Fano varieties coincide with the classical periods of their mirror Laurent polynomials. Therefore, as long as one knows the mirror Laurent polynomials, one can compute the proper Landau–Ginzburg potentials.
For example, the proper Landau–Ginzburg potentials for all Fano threefolds can be explicitly computed.
More generally, there are large databases of quantum periods for Fano manifolds to compute the proper Landau–Ginzburg potentials.

Remark

The Laurent polynomials are considered as the mirror of Fano varieties with maximal boundaries (or as the potential for the weak, non-proper, Landau–Ginzburg models). Therefore, we have an explicit relation between the proper and non-proper Landau–Ginzburg potentials.

The End

Thank you!