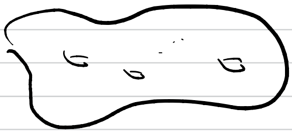
Topology of Hitchin systems — Old & New.

joint with Davesh Maulik.

Overview: Given X =  Riemann surface. $(g \geq 2)$

G reductive group.

$\rightsquigarrow$ Hitchin's integrable system $h : \mathcal{M}_G \longrightarrow \mathcal{A}_G$

(= proper Lagrangian fibration)

Want to understand: how the topology of h interacts with

- Non-abelian Hodge correspondence — The $P=W$ Conjecture '10
- Langlands duality of G — Hausel-Thaddeus topological mirror conjecture '02

How these conjectures look like ? (Rough version)

- $h : M_G \longrightarrow A_G$ $\rightsquigarrow$ encode the topology of h in a filtered vector space, called the perverse filtration $P_\bullet H^*(M_G, \mathbb{Q})$.

  X, G

- 1'st move : Change the alg. structure of M_G

  Non-abelian Hodge : $M_G \xrightarrow[\sim]{C^\infty} M_G^{Betti}$ character variety

$$H^*(M_G, \mathbb{Q}) = H^*(M_G^{Betti}, \mathbb{Q})$$

de Cataldo - Hausel - Migliorini '10

" $P = W$ " Conj.

$$P_k H^*(M_G, \mathbb{Q}) = W_{2k} H^*(M_G^{Betti}, \mathbb{Q})$$

Deligne's weight filtration ass. w/ MHS of M_G^{Betti}.

- 2'nd move : change the group $\overset{SL_n}{G} \longleftrightarrow \overset{PGL_n}{G^\vee}$ Langlands dual.

Hausel - Thaddeus '02

Topological Mirror Conj.

$$P_k H^*(M_G, \mathbb{Q}) \text{ " = " } P_k H^*_{Stringy}(M_{G^\vee}, \mathbb{Q})$$

History & Status

In order to get smooth moduli spaces, these conjectures were formulated precisely for type A.

- P = W.
  - de Cataldo - Hausel - Migliorini '10 — GL_2 ($\approx PGL_2$), SL_2, $\forall$ genus$(X) \geq 2$
  - de Cataldo - Maulik - S '19 — GL_n, genus$(X) = 2$
  - de Cataldo - Maulik - S '20 — $SL_p \Longleftrightarrow GL_p$ (p prime) $\forall$ genus$(X) \geq 2$
  - Maulik - S '22., Hausel - Mellit - Minets - Schiffmann '22 — GL_n, $\forall$ genus$(X) \geq 2$.
    ($\Rightarrow$ SL_p, $\forall$ genus$(X) \geq 2$)

- Topological Mirror Conj. (for SL_n / PGL_n)
  - Hausel - Thaddeus '02 — SL_2 / PGL_2
  - Groechenig - Wyss - Ziegler '17. — SL_n / PGL_n (p-adic integration)
    (generalized by Loeser - Wyss $\rightsquigarrow$ motivic integration)
  - Maulik - S '20. — SL_n / PGL_n sheaf-theoretic.

Perverse filtration

Input: $X \xrightarrow{f} Y$ proper map, with X, Y smooth

$\rightsquigarrow$

Output: an increasing filtration $P_0 \subset P_1 \subset \cdots \subset H^*(X, \mathbb{Q})$ governed by the topology of f.

Definition 1 (Sheaf-theoretic) $H^*(X, \mathbb{Q}) = \mathbb{H}^*(Y, Rf_* \mathbb{Q})$ $\quad Rf_*\mathbb{Q} \in D_c^b(Y)$

$$\begin{array}{ccc} \cup\!\shortmid & & \cup\!\shortmid \\ P_* & \text{"="} & \mathbb{H}^*(Y, {}^p\tau_{\le *} Rf_*\mathbb{Q}) \end{array}$$

Thm by de Cataldo – Migliorini

$\downarrow$

Definition 2 (Topological) Assume $Y = \mathbb{C}^N$ affine space.

$$P_i H^{i+k}(X, \mathbb{Q}) = \mathrm{Ker}\left(H^{i+k}(X) \to H^{i+k}(f^{-1}(\Lambda)) \right)$$

$\Lambda \subset \mathbb{C}^N$ $(k-1)$-dim'l general affine subspace

Moduli spaces (Hitchin moduli spaces)

help us to get smooth moduli.

- X Riemann surface genus ≥ 2, n rank, d. degree. $(n, d) = 1$.

- $\mathcal{M}_{GL_n}$ = moduli of slope-stable Higgs bundles $(\mathcal{E}, \theta)$
  - $\mathcal{E}$ vector bundle, rk n degree d.
  - $\theta : \mathcal{E} \to \mathcal{E} \otimes \Omega_C^1$ Higgs field ($\mathcal{O}_C$-linear)

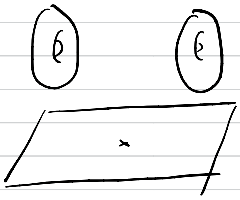
- Hitchin system $\quad h : \mathcal{M}_{GL_n} \longrightarrow \mathcal{A}_{GL_n} = \bigoplus_{i=1}^{n} H^0(C, \Omega_C^{1 \otimes i})$

  "Calculating char poly"

  $$(\mathcal{E}, \theta) \longmapsto \mathrm{char}(\theta) = (\mathrm{tr}(\wedge^i \theta))_{i=1}^{n}$$

- Example GL_1 : $\underbrace{T^* Jac_X}_{= Jac_X \times H^0(C, \Omega_C^1)} \longrightarrow \mathbb{C}^g = H^0(C, \Omega_C^1)$

  $$(\mathcal{L}, \theta : \mathcal{L} \to \mathcal{L} \otimes \Omega_C^1) \longmapsto \mathrm{char}(\theta)$$



- Non-abelian Hodge:

character variety $\qquad \pi_1(C\setminus\{p\}) \to \underset{\cup}{\overset{U(n)}{}} GL_n$

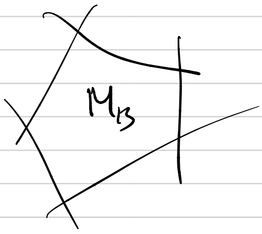
$$M_{GL_n} \overset{C^\infty}{\simeq} M^{\text{Betti}}_{GL_n} = \left\{ \begin{matrix} A_1 \cdots A_g \\ B_1 \cdots B_g \end{matrix} \in GL_n(\mathbb{C}) \;\middle|\; \overset{A_1B_1A_1^{-1}B_1^{-1}\cdots}{[A_1,B_1]\cdots[A_g,B_g]} = e^{\frac{2\pi i d}{n}} I_{d_n} \right\} /\!/_{\text{conj}}\, GL_n(\mathbb{C})$$

$$M_{GL_n} \xrightarrow{h} A_{GL_n}$$

smooth affine variety with a nontrivial mixed Hodge structure

"P=W": $\quad P_k H^*(M_{GL_n}, \mathbb{Q}) = W_{2k} H^*(M_B, \mathbb{Q})$

$\qquad\qquad (= W_{2k+1} H^*(M_B, \mathbb{Q}))$



Complexity of the topology of the Hitchin system

complexity of the "boundary" of M_B

SL_n & PGL_n moduli spaces

- Fix $\mathcal{N} \in \mathrm{Pic}^d(X)$. degree d line bundle.

$$M_{SL_n} = \{ (\mathcal{E}, \theta) \in M_{GL_n} \mid \det(\mathcal{E}) \simeq \mathcal{N},\ \mathrm{trace}(\theta) = 0 \} \subset M_{GL_n}$$

- $\Gamma := \mathrm{Pic}^0(X)[n] \circlearrowright M_{SL_n}$ $\quad \mathcal{L} \cdot (\mathcal{E}, \theta) = (\mathcal{E} \otimes \mathcal{L}, \theta)$

$\det(\mathcal{E} \otimes \mathcal{L}) \simeq \det(\mathcal{E}) \otimes \mathcal{L}^{\otimes n} \simeq \det(\mathcal{E})$

- $M_{PGL_n} := M_{SL_n} / \Gamma$, a smooth orbifold / Deligne-Mumford stack.

- Want to compare $H^*(M_{SL_n}, \mathbb{Q})$ & $H^*(M_{PGL_n}, \mathbb{Q})$

- $\Gamma \circlearrowright H^*(M_{SL_n}, \mathbb{Q})$ ⟵ $\Gamma \curvearrowright M_{SL_n}$

$$\rightsquigarrow \quad H^*(M_{SL_n}, \mathbb{Q}) = \underset{\substack{\uparrow \\ H^*(M_{PGL_n}, \mathbb{Q})}}{H^*(M_{SL_n}, \mathbb{Q})^{\Gamma}} \oplus \overbrace{\bigoplus_{\substack{0 \neq \kappa \in \hat{\Gamma} \\ \uparrow \\ \text{nontrivial} \\ \text{characters of } \Gamma}} H^*(M_{SL_n}, \mathbb{Q})_{\kappa}}^{\text{nontrivial}}$$

Refined Topological Mirror :
(Hausel – Thaddeus)

Canonical $\underset{\substack{\Psi \\ \gamma}}{\Gamma} = \underset{\substack{\Psi \\ \kappa}}{\hat{\Gamma}}$ induced by Weil pairing. (character of Γ)

$\gamma \longleftrightarrow \kappa$

$$P_k H^i(M_{SL_n}, \mathbb{Q})_{\boxed{\kappa}} \cong P_{k-D} H^{i-2D}\left(M^{\gamma}_{SL_n}, \mathbb{Q}\right)_{\kappa}$$

κ-isotypic part for SL_n = the sector of the orbifold $[M_{SL_n}/\Gamma]$ indexed by $\gamma \in \Gamma$

$\kappa \in \hat{\Gamma}$

Idea of proof (of both conjectures!)

Coh. on M_G
$\wr$
Shf on M_G
$\wr$
Shf on M_G^{mero} ←

Step 1. "Categorify" the cohomological statements.

Cohomological $\rightsquigarrow$ sheaf-theoretical

Step 2. Embed $M_{\underset{\smile}{G}}$ into a larger space of meromorphic Higgs bundles
GL_n PGL_n SL_n

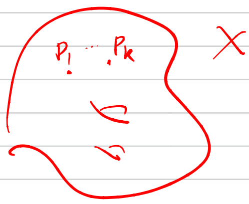
as a critical locus $M_G = \{df = 0\} \hookrightarrow M_G^{mero} \xrightarrow{f} \mathbb{C}$

Step 3. Lift the sheaf-theoretic statement of Step 1 from M_G to M_G^{mero}.

Prove this for M_G^{mero} using Support theorem. (Ngô)

Step 4. Obtain the desired statement for M_G from M_G^{mero} using vanishing cycles $\varphi_f : D_c^b(M_G^{mero}) \to D_c^b(f^{-1}(0))$.

Why M_G^{mero} is better than M_G ?



$$M_{GL_n} = \{ (\mathcal{E}, \theta) \mid \theta : \mathcal{E} \to \mathcal{E} \otimes \Omega_C^1 \} \quad + \text{ Slope stability.}$$

Pick effective $D = P_1 + \cdots + P_k$ (>0)

$$M_{GL_n}^{mero} = \{ (\mathcal{E}, \theta) \mid \theta : \mathcal{E} \to \mathcal{E} \otimes \Omega_C^1(D) \}$$

meromorphic Higgs bundles with at most simple poles at $P_1, \cdots, P_k$.

Not Symplectic!

$$M_{GL_n} \hookrightarrow M_{GL_n}^{mero}$$

Theorem

(1) [Chaudouard-Laumon] For $h^{mero} : M_{GL_n}^{mero} \longrightarrow A_{GL_n}^{mero} = \bigoplus_{i=1}^{n} H^0(C, \Omega_C^{1 \otimes i}(iD))$, every simple summand of $Rh_*^{mero} \mathbb{Q}$ has support $A_{GL_n}^{mero}$.

(2) [de Cataldo - Heinloth - Migliorini] Every Levi group of GL_n contributes the support of $Rh_* \mathbb{Q}$, for $h : M_{GL_n} \to A_{GL_n}$.

Proof of P = W

$\mathcal{U} \to C \times M_{GL_n}$ univ. bundle

▷ Reduce P = W to $C_k(\mathcal{U}) : Rh_* \underline{\mathbb{Q}} \longrightarrow Rh_* \underline{\mathbb{Q}}[2k]$

$${}^p\tau_{\leq i} \overset{?}{\longmapsto} {}^p\tau_{\leq (i-k)}$$

▷ Lift it to $M^{mero}_{GL_n}$: $C_k(\mathcal{U}^{mero}) : Rh^{mero}_* \underline{\mathbb{Q}} \longrightarrow Rh^{mero}_* \underline{\mathbb{Q}}[2k]$

$${}^p\tau_{\leq i} \overset{?}{\longmapsto} {}^p\tau_{\leq (i-k)}$$

▷ Solve the question on $M^{mero}_{GL_n}$:

(i) Global Springer theory $\rightsquigarrow$ $C_1(L) : Rh^{par}_* \underline{\mathbb{Q}} \to Rh^{par}_* \underline{\mathbb{Q}}[2]$

$${}^p\tau_{\leq i} \overset{?}{\longmapsto} {}^p\tau_{\leq (i-1)}$$

$$\Longleftrightarrow \quad C_1(L) : {}^p\mathcal{H}^i(Rh^{par}_* \underline{\mathbb{Q}}) \longrightarrow {}^p\mathcal{H}^i(Rh^{par}_* \underline{\mathbb{Q}}[2]) \text{ vanish}$$

(ii) Support thm $\rightsquigarrow$ Suffices to prove the vanishing for smooth fibers

$\rightsquigarrow$ Treat local systems.