Elliptic chiral homology and Chiral index

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Motivation: QFT and Index Theory

Topological quantum mechanics (TQM)

TQM leads to a path integral on the loop space

$$\int_{\mathrm{Map}(S^1,X)} e^{-S/\hbar} \quad \overset{\hbar\to 0}{\Longrightarrow} \quad \int_X (\text{curvatures})$$

Topological nature implies the exact semi-classical limit $\hbar \to 0$, which localizes the path integral to constant loops.

- LHS= the analytic index expressed in physics
- RHS= the topological index.

This is the physics "derivation" of **Atiyah-Singer** Index Theorem.

Algebraic Index Theorem

Given a deformation quantization $\mathcal{A}_\hbar(M) = (C^\infty(M)[[\hbar]], \star)$ on a symplectic manifold (X, ω), there exists a unique linear map

$$\mathrm{Tr} : \mathcal{A}_\hbar(M) \to \mathbb{C}((\hbar))$$

satisfying a normalization condition and the trace property

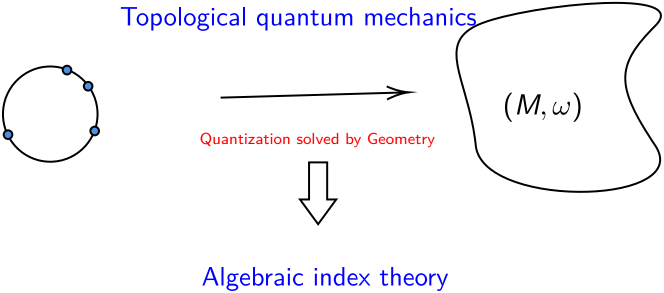
$$\mathrm{Tr}(f \star g) = \mathrm{Tr}(g \star f).$$

Then

$$\mathrm{Tr}(1) = \int_M e^{\omega_\hbar/\hbar} \hat{A}(M).$$

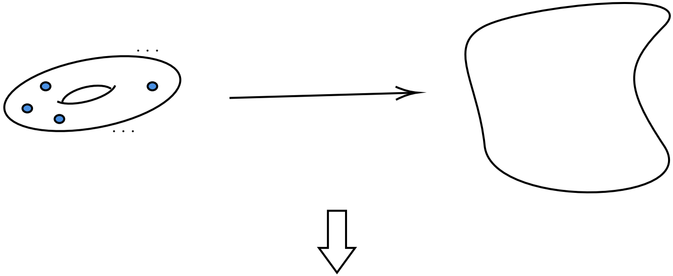
This is the algebraic index theorem which was first formulated by **Fedosov** and **Nest-Tsygan** as the algebraic analogue of **Atiyah-Singer** index theorem.

In [**Grady-Li-L 2017**, **Gui-L-Xu, 2020**], a rigorous connection between the effective BV quantization for topological quantum mechanics and the algebraic index theorem.



Witten's "Index Theorem" on loop space

Replace S^1 by an elliptic curve E. (**Witten**: index of Dirac operators on loop space).



2d Chiral analogue of algebraic index?

Observables and Factorization algebras

A QFT is usually described by a manifold X and the data of fields

$$\text{Spacetime} : X \quad \Longrightarrow \quad \text{Fields} : \mathcal{E} = \Gamma(X, E).$$

One algebraic structure associated to the topology of X is

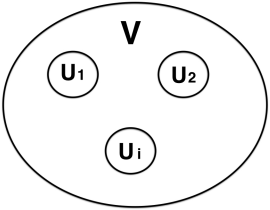
observables=functions on fields

Given an open subset $U \subset X$, we can talk about

$$Obs(U) = \text{observables supported in U}$$

Example: δ-function.

Observables form an algebraic structure as follows: given disjoint open subset U_i contained in an open V: $\coprod_i U_i \subset V$



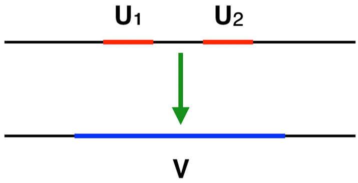
we have a factorization product for observables

$$\bigotimes_i Obs(U_i) \to Obs(V).$$

- Physics: OPE (operator product expansion)
- Factorization algebra: **Beilinson-Drinfeld** in 2d CFT
- Realization in perturbative QFT: **Costello-Gwilliam**

Example: $\dim X = 1$ (topological quantum mechanics)

QFT in $\dim = 1$ is quantum mechanics.



In the topological case, for any contractible open U, $Obs(U) = A$.
The factorization product doesn't depend on the location and size:

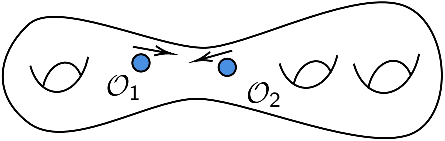
$$A \otimes A \to A.$$

We find an (homotopy) associative algebra.

Example: $\dim X = 2$ (chiral conformal field theory)

The factorization product of 2d chiral theory is holomorphic.

$$\mathcal{O}_1(z)\mathcal{O}_2(w) \sim \sum_n \frac{\mathcal{O}_{1(n)}\mathcal{O}_2(w)}{(z-w)^{n+1}}$$



which is the 2d analogue of "associative product". We find ∞-many binary operations $\mathcal{O}_{1(n)} \cdot \mathcal{O}_2$!

In this case, observable algebra forms a vertex algebra.

An important class of quantities are correlation functions of observables. They capture "global" information of the theory.

- Local correlation

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_i(x_i)\cdots\mathcal{O}_n(x_n)\rangle\,, \quad x_i \in X.$$

  It is singular when points collide, hence a function on

$$\mathrm{Conf}_n(X) := \{x_1,\cdots,x_n \in X | x_i \neq x_j \text{ for} i \neq j\}.$$

- Many interesting non-local information is hidden in

$$\int_{\mathcal{Z}\subset \mathrm{Conf}_n(X)} \langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_i(x_i)\cdots\mathcal{O}_n(x_n)\rangle$$

  which might be divergent and require further renormalization.

Batalin-Vilkovisky (BV) Quantization formalism

Homological methods (such as BRST-BV) arise in physics as a general method to quantize theories with gauge symmetries.

BV algebra

A **Batalin-Vilkovisky** (BV) algebra is a pair $(\mathcal{A}, \Delta)$ where

- $\mathcal{A}$ is a $\mathbb{Z}$-graded commutative associative unital algebra.
- $\Delta : \mathcal{A} \to \mathcal{A}$ is a linear operator of degree 1 such that $\Delta^2 = 0$.
- The **BV bracket** $\{-,-\} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ by

$$\{a, b\} := \Delta(ab) - (\Delta a)b - (-1)^{|a|} a\Delta b, \ \ a, b \in \mathcal{A}.$$

  $\{-,-\}$ satisfies a version of graded Leibnitz rule.

Example (Polyvector fields)

The space of smooth polyvector fields with a divergence operator

$$(\mathrm{PV}^\bullet(X) = \Gamma(X, \wedge^\bullet T_X), \quad \Delta = \text{divergence})$$

is a BV algebra.

Roughly speaking, BV quantization in QFT leads to

- Quantum algebra Obs of observables.
- $(C_\bullet(\mathrm{Obs}), d)$: a chain complex via algebraic structures of Obs.
- A BV algebra $(\mathcal{A}, \Delta)$ ("zero modes") with $\int_{BV} : \mathcal{A} \to \mathbb{C}$.
- A $\mathbb{C}[[\hbar]]$-linear map ("integrating out modes")

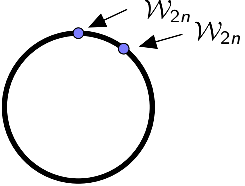
  $$\langle - \rangle : C_\bullet(\mathrm{Obs}) \to \mathcal{A}((\hbar))$$

  satisfying quantum master equation

  $$(d + \hbar\Delta)\langle - \rangle = 0. \qquad \text{(QME)}$$

  This means it is a chain map intertwining d and $\hbar\Delta$.
- Partition function: $\mathrm{Index} = \int_{BV} \langle 1 \rangle$.

Example: Topological Quantum Mechanics (TQM)



- Fields: $\varphi \in \Omega^\bullet(S^1) \otimes \mathbb{R}^{2n}$. So $\varphi : S^1_{dR} \to \mathbb{R}^{2n}$.
- Local observables: Weyl algebra

$$\mathrm{Obs}_{1d} = \mathcal{W}_{2n} = \left(\mathbb{C}[\![p_i, q^i]\!][\![\hbar]\!], \star\right)$$

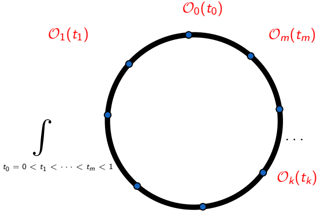
- $(C_\bullet(\mathrm{Obs}_{1d}), b) =$ the Hochschild chain complex.
- BV algebra $(\mathcal{A}_{1d}, \Delta) = (\widehat{\Omega}^\bullet(\mathbb{R}^{2n}), \mathcal{L}_\Pi)$. Here $\Pi =$ Poisson tensor. In physics, this describes the geometry of zero modes.

$$\mathcal{A}_{1d} = \text{functions on } H^\bullet(S^1) \otimes \mathbb{R}^{2n}$$

- $\langle - \rangle_{1d} : C_\bullet(\mathcal{W}_{2n}) \to \mathcal{A}_{1d}(\!(\hbar)\!)$ where

$$\langle \mathcal{O}_0 \otimes \mathcal{O}_1 \cdots \otimes \mathcal{O}_m \rangle_{1d} \qquad\qquad \mathcal{O}_i \in \mathcal{W}_{2n}$$

$$= \int_{t_0=0<t_1<\cdots<t_m<1} \left\langle \mathcal{O}_0(\varphi(t_0))\mathcal{O}_1^{(1)}(\varphi(t_1)) \cdots \mathcal{O}_m^{(1)}(\varphi(t_m)) \right\rangle_{free}$$

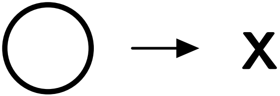


It satisfies

$$\text{QME} \qquad (b + \hbar\Delta)\langle - \rangle_{1d} = 0$$

Here b is the Hochschild differential.

Ref: [**Gui-L-Xu**, 2020]



These data glues [**Fedosov**] to give a Weyl bundle $\mathcal{W}(X) \to X$.

- [**Grady-Li-L**]: BV quantum master equation is equivalent to Fedosov's flat connection on $\mathcal{W}(X)$.

"Quantization is solved by Geometry".

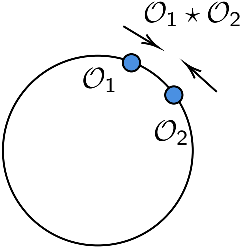
- $\langle - \rangle_{1d}$ leads to a trace map on deformation quantized algebra, equivalent to the formula by [**Feigin-Felder-Shoikhet**].
- [**Grady-Li-L**, **Gui-L-Xu**]: BV quantization of TQM gives

$$\int_{BV} \langle 1 \rangle_{1d} = \int_X e^{\omega_\hbar/\hbar} \hat{A}(X).$$

2d Chiral Conformal Field Theory

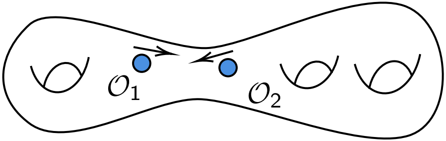
1d TQM	2d Chiral CFT
S^1	Σ
Associative algebra	Vertex operator algebra

Associative product



Operator product expansion

$$\mathcal{O}_1(z)\mathcal{O}_2(w) \sim \sum_n \frac{\mathcal{O}_{1(n)}\mathcal{O}_2(w)}{(z-w)^{n+1}}$$



A chiral σ-model

$$\varphi : \Sigma \to X$$

will produce a bundle $\mathcal{V}(X)$ of chiral vertex operator algebras

$$\begin{array}{c} \mathcal{V}(X) \\ \big\downarrow \\ X \end{array}$$

This is the chiral analogue of Weyl bundle in TQM.

Theorem (L)

The BV quantization of the 2d chiral model is equivalent to solving a flat connection on the vertex algebra bundle $\mathcal{V}(X)$

$$D = d + \frac{1}{\hbar}\left[\oint \mathcal{L}, -\right], \quad D^2 = 0$$

where $\mathcal{L} \in \Omega^1(X, \mathcal{V}(X))$ and $\oint \mathcal{L}$ is the associated chiral vertex operator fiberwise.

- This is the chiral analogue of Fedosov connection. "Quantization is again solved by Geometry".
- BRST reduction of chiral models falls into this setup

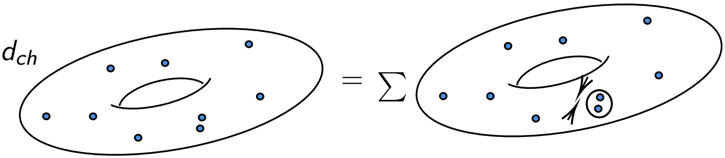
$$\oint \mathcal{L} = \text{BRST operator.}$$

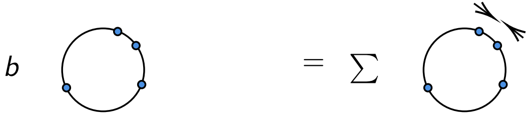
Ref: [**L**] Vertex algebras and quantum master equation. *To appear in JDG.*

Elliptic chiral homology

- In [**Zhu**, 1994], **Zhu** studied the space of genus 1 conformal block (the 0-th elliptic chiral homology) and establish the modular invariance for certain class of VOA.
- **Beilinson** and **Drinfeld** define the chiral homology for general algebraic curves using the Chevalley-Cousin complex.
- Recently, [**Ekeren-Heluani**,2018,2021]: an explicit complex expressing the 0th and 1st elliptic chiral homology.

Intuitively, the chiral differential in the chiral complex looks like a 2d chiral analogue of the Hochschild differential b.





$$b(a_0 \otimes \cdots \otimes a_p)$$

$$= (-1)^p a_p a_0 \otimes \cdots \otimes a_{p-1} + \sum_{i=0}^{p-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_p.$$

We briefly review the construction of **Beilinson** and **Drinfeld**.

- $\mathcal{M}(X)$: category of (right) $\mathcal{D}$-modules on $X = \Sigma$
- $\mathcal{M}(X^{\mathcal{S}})$: category of (right) $\mathcal{D}$-modules on $X^{\mathcal{S}}$

  $M \in \mathcal{M}(X^{\mathcal{S}})$ is rule that assigns to each finite index set $I \in \mathcal{S}$

  $$\text{a right } \mathcal{D} - \text{module } M_{X^I} \text{ on } X^I.$$

  (satisfying some compatibility conditions.)
- There is an exact fully faithful embedding

  $$\Delta_*^{(S)} : \mathcal{M}(X) \hookrightarrow \mathcal{M}(X^{\mathcal{S}})$$

  defined by $(\Delta_*^{(S)} M)_{X^I} := \Delta_*^{(I)} M$, where $\Delta^{(I)} : X \hookrightarrow X^I$.

The category $\mathcal{M}(X^{\mathcal{S}})$ carries a tensor structure $\otimes^{\mathrm{ch}}$ and a chiral algebra $\mathcal{A}$ is a Lie algebra object via $\Delta_*^{\mathcal{S}}$.

We consider the Chevalley-Eilenberg complex

$$(C(\mathcal{A}), d_{\mathrm{CE}}) = (\oplus \mathrm{Sym}^{\bullet}_{\otimes^{\mathrm{ch}}}(\Delta_*^{(S)}\mathcal{A}[1]), d_{\mathrm{CE}}),$$

which is a complex in $\mathcal{M}(X^{\mathcal{S}})$.

The chiral homology (complex) $C^{\mathrm{ch}}(X, \mathcal{A})$ is defined by $R\Gamma_{\mathrm{DR}}(X^{\mathcal{S}}, C(\mathcal{A}))$, where

Dolbeault Resolution Chevalley-Eilenberg

$$R\Gamma_{DR}\left(X^{\mathcal{S}}, C(\mathcal{A})\right)$$

Spencer Resolution

Example: $\beta\gamma - bc$ system

The VOA $\mathcal{V}^{\beta\gamma-bc}$ of $\beta\gamma - bc$ system is the chiral analogue of Weyl/Clifford algebra.

$$\beta(z)\gamma(w) \sim \frac{1}{z-w} \qquad b(z)c(w) \sim \frac{1}{z-w}.$$

It gives rise to a chiral algebra (in the sense of Beilinson and Drinfeld) $\mathcal{A}^{\beta\gamma-bc} = \mathcal{V}^{\beta\gamma-bc} \otimes_{\mathcal{O}_x} \omega_X$ on a Riemann surface $X = \Sigma$.

The factorization homology (complex)

$$(C_\bullet(\mathcal{V}^{\beta\gamma-bc}(\mathbf{h})), d_{ch}) \quad \text{in the BV formalism}$$

will be the chiral chain complex $C^{\mathrm{ch}}(X, \mathcal{A}^{\beta\gamma-bc})$.

Theorem (Gui-L)

Let X be an elliptic curve E_τ. We can construct an explicit map

$$\langle - \rangle_{2d} : C^{\mathrm{ch}}(E_\tau, \mathcal{A}^{\beta\gamma - bc}) \to \mathcal{A}_{2d}((\hbar))$$

satisfying

$$QME: \quad (d_{\mathrm{ch}} + \hbar\Delta)\langle - \rangle_{2d} = 0.$$

Roughly speaking, this map is defined by

$$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{2d} := \fint_{E_\tau^n} \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle.$$

- $\mathcal{A}_{2d}$ are functions on zero modes (=copies of $H^\bullet(E_\tau, \mathcal{O}_{E_\tau})$).
- $\langle - \rangle_{2d}$ is a quasi-isomorphism.
- $\langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle$ is local correlation (via Feynman rules).
- $\fint$ is the regularized integral introduced by [**L-Zhou**]. This is a geometric renormalization method for 2d chiral QFT.
- The BV trace map leads to Witten genus.

The issue of singular integral and renormalization

We need to understand the integral of local correlators

$$\int_{\Sigma^n} \langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\rangle \text{''} \overset{?}{=} \text{''}$$

Unlike the situation in topological field theory, $\langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\rangle$ is very singular along diagonals and there is no way to extend it to certain compactification of $\mathrm{Conf}_n(\Sigma)$.

Regularized integral (L-Zhou 2020)

Let us first consider the integral of a 2-form ω on Σ with meromorphic poles of arbitrary orders along a finite subset $D \subset \Sigma$. Locally we can write $\omega = \frac{\eta}{z^n}$ where η is smooth 2-form and $n \in \mathbb{Z}$.

We can decompose ω into

$$\omega = \alpha + \partial\beta$$

where α is a 2-form with at most logarithmic pole along D, β is a $(0,1)$-form with arbitrary order of poles along D, and $\partial = dz\frac{\partial}{\partial z}$ is the holomorphic de Rham. We define the regularized integral

$$\boxed{⨍_\Sigma \omega := \int_\Sigma \alpha + \int_{\partial\Sigma} \beta}$$

This does not depend on the choice of the decomposition.

$\fint_\Sigma$ is invariant under conformal transformations. The conformal geometry of Σ gives an intrinsic regularization of the integral $\int_\Sigma \omega$.

The regularized integral can be viewed as a "homological integration" by the holomorphic de Rham ∂

$$\fint_\Sigma \partial(-) = \int_{\partial\Sigma}(-).$$

The $\bar\partial$-operator intertwines the residue

$$\fint_\Sigma \bar\partial(-) = \mathrm{Res}(-).$$

In general, we can define

$$\fint_{\Sigma^n}(-) := \fint_{\Sigma}\fint_{\Sigma}\cdots\fint_{\Sigma}(-)\,.$$

This gives a rigorous and intrinsic definition of

$$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{2d} := \fint_{\Sigma^n} \langle \mathcal{O}_1(z_1)\cdots \mathcal{O}_n(z_n)\rangle.$$

It exhibits all the required properties:

- Holomorphic Anomaly Equation. (**L-Zhou**, arXiv:2205.14562)
- Contact equations. (**Gui-L-Tang**, in preparation)
- $\ldots$

Elliptic chiral index (after Douglas-Dijkgraaf)

The partition function of a chiral deformation by a chiral lagrangian $\mathcal{L}$ is given by

$$\left\langle e^{\frac{1}{\hbar}\int_\Sigma \mathcal{L}} \right\rangle_{2d}.$$

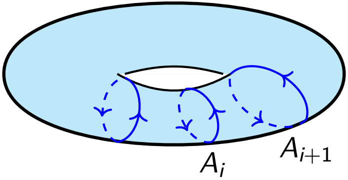
If we quantize the theory on elliptic curve $\Sigma = E_\tau$,

$$\boxed{\lim_{\bar\tau\to\infty} \left\langle e^{\frac{1}{\hbar}\int_{E_\tau} \mathcal{L}} \right\rangle_{2d} = \mathrm{Tr}_{\mathcal{H}}\, q^{L_0 - \frac{c}{24}} e^{\frac{1}{\hbar}\oint dz\mathcal{L}}, \quad q = e^{2\pi i\tau}}$$

where the operation $\lim_{\bar\tau\to 0}$ sends

almost holomorphic modular forms $\Longrightarrow$ quasi-modular forms.

This can be viewed as a chiral algebraic index on the loop space.
The regularized integral [**L-Zhou**] precisely explains $\bar\tau \to \infty$.



Theorem (L-Zhou 2020)

Let $\Phi(z_1, \cdots, z_n; \tau)$ be a meromorphic elliptic function on $\mathbb{C}^n \times \mathbf{H}$ which is holomorphic away from diagonals. Let $A_1, \cdots, A_n$ be n disjoint A-cycles on E_τ. Then the regularized integral

$$\fint_{E_\tau^n} \left(\prod_{i=1}^{n} \frac{d^2 z_i}{\operatorname{im} \tau} \right) \Phi(z_1, \cdots, z_n; \tau) \quad \textit{lies in} \quad \mathcal{O}_{\mathbf{H}}[\frac{1}{\operatorname{im} \tau}] \quad \textit{and}$$

$$\boxed{\lim_{\bar{\tau} \to \infty} \fint_{E_\tau^n} \left(\prod_{i=1}^{n} \frac{d^2 z_i}{\operatorname{im} \tau} \right) \Phi(z_1, \cdots, z_n; \tau) = \frac{1}{n!} \sum_{\sigma \in S_n} \int_{A_1} dz_{\sigma(1)} \cdots \int_{A_n} dz_{\sigma(n)} \Phi(z_1, \cdots, z_n; \tau)}.$$

In particular, $\fint_{E_\tau^n}$ gives a geometric modular completion for quasi-modular forms arising from A-cycle integrals.

Algebraic Index vs Elliptic Chiral Index

1d TQM	2d Chiral CFT
Associative algebra	Vertex operator algebra
Hochschild homology	Chiral homology
QME: $(\hbar\Delta + b)\langle - \rangle_{1d} = 0$	QME: $(\hbar\Delta + d_{ch})\langle - \rangle_{2d} = 0$
$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{1d} =$integrals on the compactified configuration spaces of S^1	$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{2d} =$regularized integrals of singular forms on Σ^n
Algebraic Index theory	Elliptic Chiral Algebraic Index

Joint work with **Zhengping Gui**. arXiv:2112.14572 [math.QA]

Thank you!