10-25-22

The Chromatic Lagrangian: Wavefunctions & OGW conjs.

w/ Schrader, Shen

website

Idea : 1. Generalize Aganagic-Vafa to

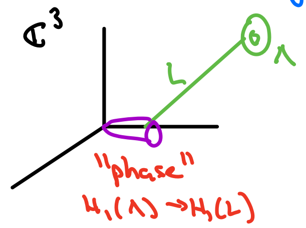
- new geometries
- higher genus

2. Exploit cluster theory

Result is a groupoid capturing calculation scheme,
plus accompanying geometric interpretations.

[Córdova-Espahbodi-Haghighat-Rastogi-Vafa, Dimofte-Gabella-Goncharov, Luo-Zhu, Kucharski-Reineke-Stosic-Sulkowski]

Review of Aganagic-Vafa (AKV, ADKMV)

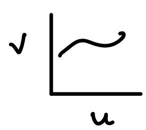


Recover disk invts from moduli of brane in good coords

Use M.S.

- $\mathbb{C}^3 \leftrightarrow ab = -1 + X + Y$ — Torus moduli
- brane $\leftrightarrow b = 0$
- moduli $X + Y = 1$
- coords $X = e^{u}$ fr $Y = e^{-v}$ ph
- disk pot'l $v = \partial_u W$
  $= -\ln(1-X)$

$W = \mathrm{Li}_2(X)$



$v = dW$
W coord dep!
framing $u \to u + pv$

Framing $X \to XY^{1-p}$, $Y \to Y$

$W \to \dots$ $\sum n_d \, \mathrm{Li}_2(X^d)$ O-V integrality ($g=0$)

Wavefunction: $\Psi \sim e^{W/\hbar}$ should encode higher genus

Quantization: u, v conj vars (v " $-i\hbar\partial_u$") (from dual CS thy ADKMV) $YX = qXY$ $q = e^{i\hbar}$

$-v$ translation $u \to u + \hbar$ $\Rightarrow Y\Psi = \Psi(qX)$

Quantized moduli eq. $(-1 + X + Y)\Psi = 0$
$\Psi(qX) = (1-X)\Psi(X)$

$\Psi(X) = \frac{1}{1-X}\,\frac{1}{1-qX}\cdots = \Phi(X)$
q-dilog

O-V integrality (all g)

Approach used for : knot conormals (other tori) AENV
other geometries (TCY3) AKV, Marino

We study 

, Lags w/ Legendrian surf bdry

So we must

- construct Legendrian surfs, Lag fillings
- find their moduli space* (inside a torus)
- quantize
- clarify notion of framing (coords)
- find wavefunction*

... then we can

- check integrality

... and

- make OGW predictions

Rmk: impossible to compute

*How? Clusters! Sheaves!

Clusters: Fock - Goncharov

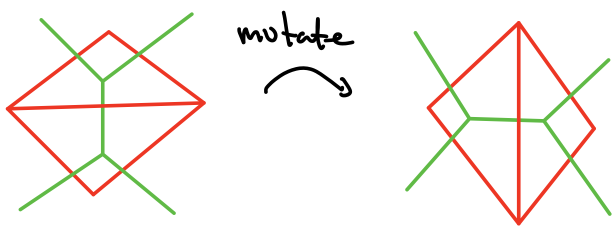
$\mathcal{X}_{G,S}$ Poisson vty whose points are G-local systems on decorated surface S with monodromy-invariant Borel (pt in $\mathbb{P}^1$) @ pts

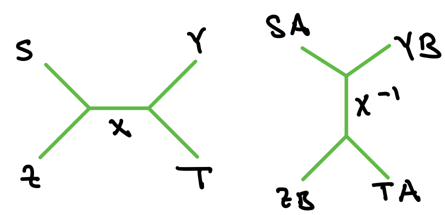
$G = PGL_2$, $S = S^2 \setminus$ pts

Cluster vty w/ toric charts ("seeds")

Charts here = Δ^ns $\longleftrightarrow$ graphs Γ

coord for each edge





$A = 1 + qX$

$B = (1 + q X^{-1})^{-1}$

- Cluster $\mathcal{X}$ f^n compatible w/ quant^{n} [F-G]

eiv : $\mathcal{X}_{G,S} \longrightarrow H^{\#\text{punctures}}$ $\qquad \mathcal{X}^{un} := \text{eiv}^{-1}(1)$

$\mathcal{X}_{G,S}$ ↺ antisymp involution, i (reflection) $\qquad i : h_k \longmapsto h_k^{-1}$

Fixed pt set LAG. Component $\mathcal{M}$

*

Structure like $\mathcal{X}^{un}$ symp $\qquad$ CHART: T_Γ^q $\quad$ q-torus

A-V : $\mathcal{M}$ Lag $\qquad \cup$

(triv loc syst) $\qquad \mathcal{M}_\Gamma$ $\quad$ ideal

But what does this have to do with branes and moduli?

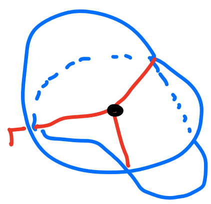
Spoiler: 1) cubic graphs $\rightsquigarrow$ Λ, Legendrian b.c.'s for Lag objs

2) $\mathcal{M}$ moduli of $Sh^1_\Lambda(\mathbb{R}^3) \cong Fuk_\Lambda(T^*\mathbb{R}^3)$

3) chart $\overset{\text{non can}}{\cong}$ "$Loc(\Lambda)$" $\cong (\mathbb{C}^\times)^{b_1(L)}$

4) In chart Λ, $\mathcal{M}_\Lambda \subset (\mathbb{C}^\times)^{b_1(L)}$ generalizes AV

1) $\Gamma \rightsquigarrow \Lambda$



2-sheeted front of Λ near vertex •

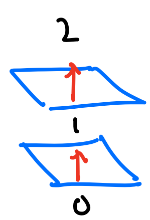
$y_i = \frac{\partial z}{\partial x_i}$ $\quad \Lambda \subset S^2 \times \mathbb{R} \subset \mathbb{R}^3$

$\Lambda \xrightarrow{2:1} S^2$ branched over $Vert(\Gamma)$

2) $\mathcal{M}$: $Sh^1_\Lambda(\mathbb{R}^3)$ init cat (GKS)

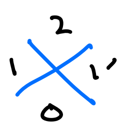
$\cong Fuk^1_\Lambda(T^*\mathbb{R}^3)$

Objs $\Rightarrow \mathcal{M}_\Lambda$ $\leftarrow$ over generic pt of S^2, sheaf



Flag in $\mathbb{C}^2$

$\leftrightarrow$ pt p in $\mathbb{P}^1$



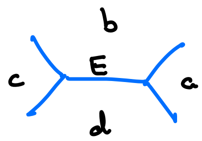
$p \neq p'$

$$\mathcal{M}_\Lambda \cong \left\{ \begin{matrix} \text{map colorings of } \Gamma \\ \text{colors in } \mathbb{P}^1 \end{matrix} \right\} \Big/ PGL_2$$

$\cong$ TRIV LOC SYST $\overset{\text{Borel}}{\subset}$ $\mathcal{X}^{un}_{PGL_2, S^2}$

3) Chart   Edge of Γ connects branch pts $\Rightarrow$ loop of Λ

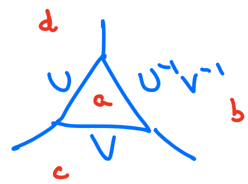
$$\mathcal{M} \longrightarrow \text{"Loc}(\Lambda)\text{"}$$



$\mapsto$ Hol(E) $= -X(a,b,c,d)$

(F-G $\Rightarrow$ compatible w/ X mut^{n}
TZ, STWZ
(and quantization))

4)   Equations for $\mathcal{M}$



compute : prod edges $= 1$
(imposed by U, V)

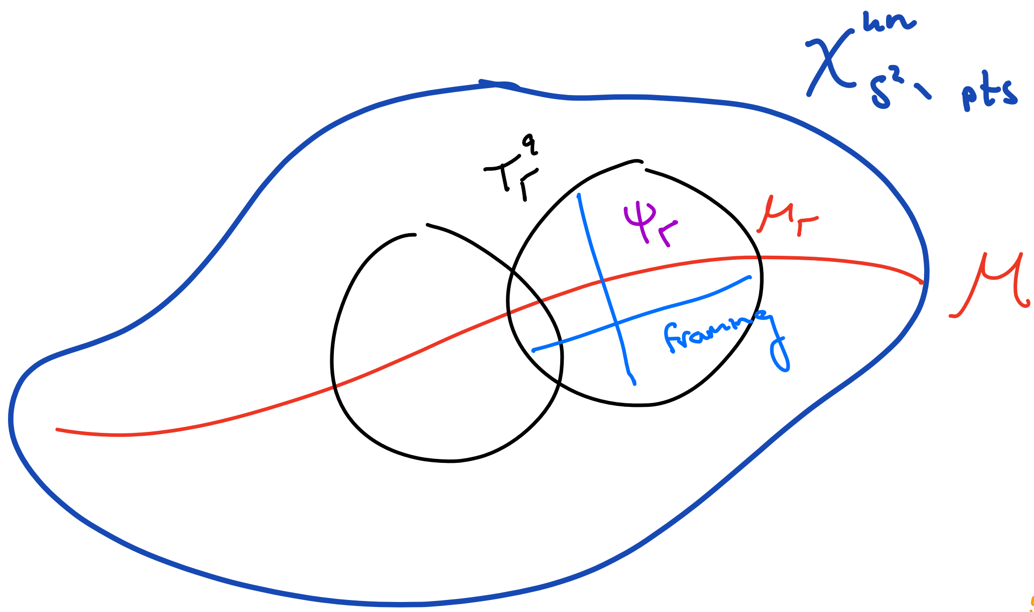
- one for each face

compute $1 + V + UV = 0$

Frame/change
Coord UV^{-1}
$-1 + U + V \Rightarrow \psi(qX) = (1-X)\psi(X)$

$\psi = \frac{1}{1-X}\frac{1}{1-qX}\cdots$

RECAP: WE HAVE A LEGENDRIAN AND MODULI SPACE OF SHEAVES INSIDE EACH CLUSTER CHART, CUT OUT BY EQUATIONS



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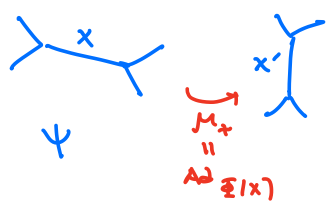
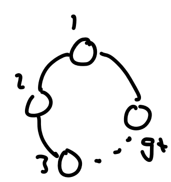
How to compute? a) mutate



$\Psi' = \Phi(x)\Psi$ ✓

b) change frame   KS formula ✓

$\Rightarrow$ Many wavefunctions from $\Psi_{\text{neck}} \equiv 1$

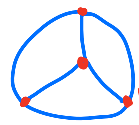


What is the geometry?

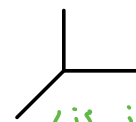
Choice of phase $\Rightarrow$ Lag $L \subset \mathbb{C}^3$, $\partial L = \Lambda$ $\Rightarrow H_1(\Lambda) \to H_1(L)$

Wavefunction $\Rightarrow$ OGW

How? A-V brane



Compute: (1) T^2

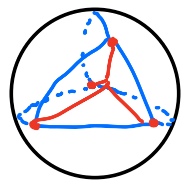


$(e^{is}, e^{it}, e^{-i(s+t)})$

$\text{Re} \downarrow$

$S^2 \times \mathbb{R}$

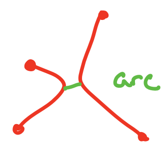
Conical filling



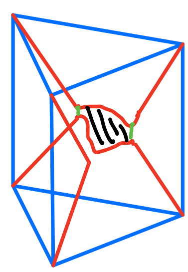
Cone(T^2)

Smoothing (H-L) L

(phase)



Embeddings, relations



smooth filling of genus 2

no loops in tangle

Combinatorics

$$\begin{tikzcd}
\mathbb{Z}^E / R_{F_E} \arrow[r] \arrow[d, hook] & H_1(\Lambda) \arrow[d, hook] \\
\mathbb{Z}^{A \cup E} / R_{F_I} \cup R_{F_E} \arrow[r] & H_1(L)
\end{tikzcd}$$

Then splitting $H_1(\Lambda) \underset{\text{framing}}{\overset{\text{phase}}{\rightleftarrows}} H_1(L)$ frames $H_1(\Lambda) \cong T^* H_1(L)$ + coords

$\Rightarrow$ coords for $\mathcal{M}$ $\Rightarrow$ OGW

Rmk: diff't isotropic framings $\Leftrightarrow$ $g \times g$ symm, int matx A.

Rmk:

"Framing Duality"

$$\Psi^{(A)}_{CANOE_g} \cong DT_{Q^A}$$

conj. $\wr\wr$

$$\cong OGW^{(A)}_{CLIFF_g}$$

$$\sum_{\substack{d \in \mathbb{Z}^g_{\geq 0} \setminus \{0\} \\ s \in \mathbb{Z}}} (-q^2)^s x^d \, n_{d,s}$$

$$n_{d,s} = \dim H^{2s}(M_d)$$

$$\mathbb{C}^N / GL_d$$

Guo-Zhu: ☮

Structure

Framed seed groupoid

OBJ $\underline{i}$: a_{ij} antisymm pairing on lattice P

$\underline{P} = P/P_c$ symp

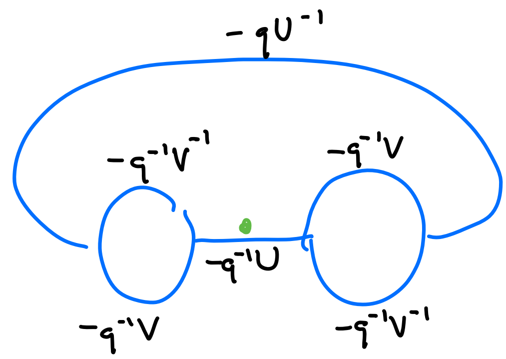
polarization $0 \to K \to \underline{P} \to K^\vee \to 0$

framing $f : T^q_{\underline{P}} \xrightarrow{\sim} D_{2g}$

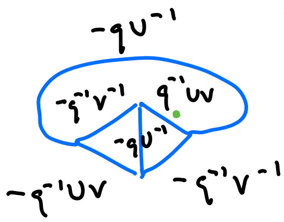
HOMS:
- frame change
- mutation

Ex

$\underline{i}$ $\quad 1$



$VU = q^2 UV$



$\underline{j}$

$(X;q^2)_\infty$

$1 + UV - V = 0$

frame change

$\underline{k}$ $\quad (X;q^2)_\infty^{-1}$

Then: $\underline{L}_0 \rightsquigarrow \underline{i}$ through admissible seq. of mutations

$\Rightarrow \psi_{\underline{i}}$ well-defined